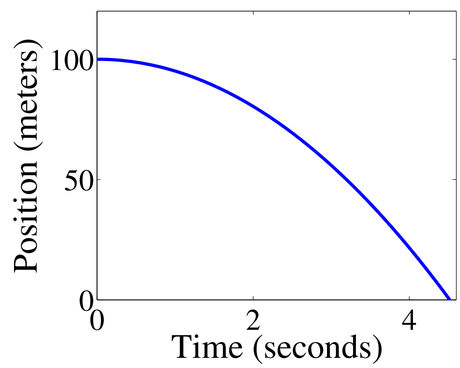
Position, velocity, and acceleration

The standard example we use when discussing instantaneous rates of change are the relationships between position, velocity, and acceleration.

Define $s(t)$ as the position, or displacement: it's a function that describes movement along a line. For example, consider a troublesome monkey in ball dropped from a height of 100 m. Its position in meters, relative to the earth's surface, can be modeled by

$$s(t) = 100 - 4.9t^2.$$

We can also sketch the curve giving $s(t)$, which in this case represents how high off the ground the ball is at time t.



The ball hits the ground at time τ when the height is zero, that is, $s(\tau) = 0$. Therefore to find τ, solve $s(\tau) = 0$:

$$\begin{aligned} s(\tau) &= 0 \\ 100 - 4.9\tau^2 &= 0 \\ \tau &\approx 4.5 \text{ seconds.} \end{aligned}$$

We'll use this time to calculate the average trip velocity.

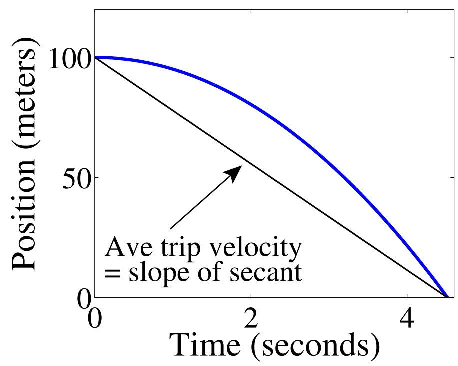
The rate of change of position is the velocity. The units are in distance/time (e.g. miles per hour). The ball's velocity can be obtained from the position. Its average speed over an interval $t_0 \leq t \leq t_1$ is given by

$$\text{average velocity between } t_0 \leq t \leq t_1 = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

This average is the slope of the secant line connecting $(t_0, s(t_0))$ and $(t_1, s(t_1))$. So for our example, the average speed between the time the ball was dropped and when it hits the ground (at time $\tau \approx 4.5$ s, calculated above) is

$$\begin{aligned} \text{average velocity between } 0 \leq t \leq \tau &= \frac{s(\tau) - s(0)}{\tau - 0} \\ &= \frac{0 - 100 \text{ m}}{4.52 \text{ s}} \\ &= -22.1 \text{ m/s.} \end{aligned}$$

Again, this is the slope of the secant line connecting the start and end of the trip, shown as the black curve in the graph:



To compute the average velocity we used the fact that $s(\tau) = 0$ – the trip ends when the ball hits the ground, so its height is 0 – and $s(0) = 100$ – the ball is dropped at $t = 0$ from a height of 100m. Alternatively you can plug in the formula and numbers to get the same result.

We can write this average velocity over the time interval $t_0 \leq t \leq t_1$ as $t_0 \leq t \leq t_0 + h$ (the previous calculation would therefore have $t_0 = 0$ and $h = \tau$); then the average velocity becomes

$$\text{average velocity between } t_0 \leq t \leq t_0 + h = \frac{s(t_0 + h) - s(t_0)}{h},$$

which we recognize as the difference quotient.

We can calculate the *instantaneous* velocity at time t_0, $v(t_0)$, by taking the time increment in the average velocity calculation $h \to 0$:

$$v(t_0) = \lim_{t \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

The result is the slope of the tangent of the position curve at t_0, and the **derivative** of the position $s(t)$ at time t_0. Examples of these tangent lines are shown in the next figure. Therefore we can write

$$v(t_0) = s'(t_0),$$

that is, the velocity at time t_0 is the derivative of the position $s(t)$ at time t_0. If time t is arbitrary,

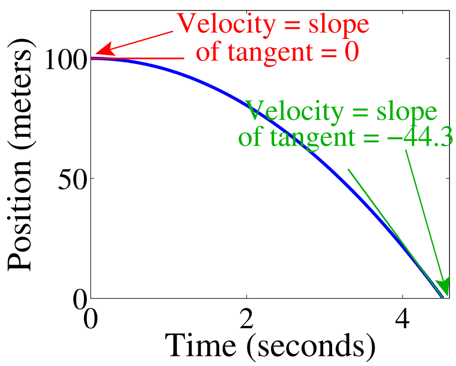
$$v(t) = s'(t)$$

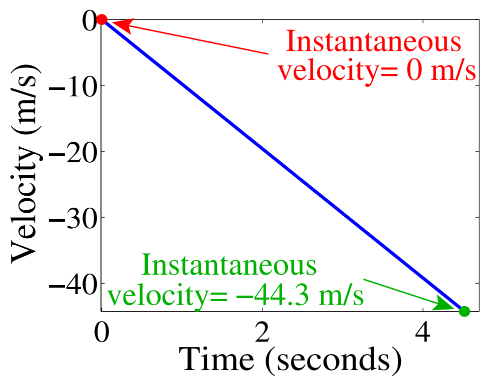
that is, the derivative of position $s(t)$ gives the velocity as a function of time, $v(t)$.

Going back to the dropped ball example, we can use this derivative to evaluate the velocity of the ball when it is dropped and when it hits the ground. The ball's position is given by $s(t) = 100 - 4.9t^2$. Its velocity function is therefore

$$\begin{aligned}
v(t) &= s'(t) \\
&= \frac{d}{dt}\left(100 - 4.9t^2\right) \\
&= \frac{d}{dt}(100) - \frac{d}{dt}\left(4.9t^2\right) \quad \text{sum/difference rule} \\
&= 0 - 9.8t \quad \text{power rule + derivative of a const. is 0} \\
v(t) &= -9.8t.
\end{aligned}$$

Thus $v(t) = -9.8t$ is the speed of the ball, in meters per second (time t is given in seconds). The ball is dropped at time $t = 0$. Its instantaneous velocity when it is dropped is therefore 0 m/s. Which makes sense: the ball is dropped, not thrown! Then the velocity of the ball as it hits the ground (at time $\tau = 4.5$ s) is approximately $v(4.5)$=-9.8(4.5)=-44.3 m/s. Graphically we show that these velocities correspond to the slope of the tangent at time $t = 0$ and when the ball hits the ground:





Notice that the velocity $v(t)$ is negative, because we assume down is the negative direction. Notice also that the ball is going much faster when it hits the ground than it was when it was dropped: it accelerated on its way down.

Acceleration $a(t)$ is defined as the change in velocity over time. Its units are in distance/time^2. Just as we did with position and velocity, we can characterize the average rate of change in velocity, i.e. the acceleration, in a time interval, say $t_0 \le t \le t_0 + h$, as

$$\text{average acceleration over } t_0 \le t \le t_0 + h = \frac{v(t_0 + h) - v(t_0)}{h}.$$

The *instantaneous acceleration* at time t_0 $a(t_0)$ (instantaneous rate of change in velocity) is realized when you take the limit of this expression as $h \to 0$, that is, you shrink the interval over which you take the average, to zero,

$$a(t_0) = \lim_{h \to 0} \frac{v(t_0 + h) - v(t_0)}{h}.$$

It's the tangent of the velocity function! But this limit is the definition of the derivative of the velocity at time $t = t_0$,

$$a(t_0) = v'(t_0).$$

If time t is arbitrary,

$$a(t) = v'(t)$$

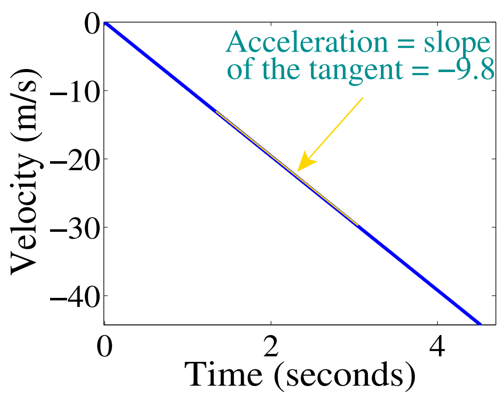
that is, the derivative of velocity $s(t)$ gives the acceleration as a function of time, $a(t)$.

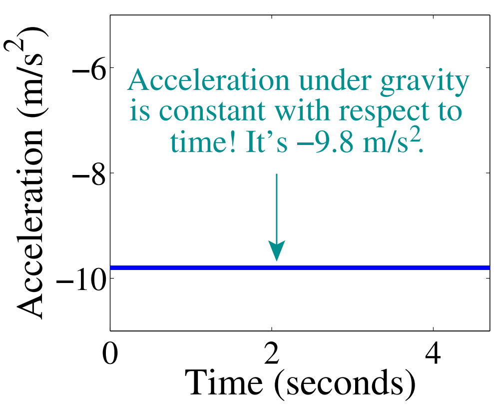
Back to the ball dropping example. What is the ball's acceleration when it is dropped? What is its acceleration when it hits the ground? The acceleration function is given by

$$a(t) \quad = \quad v'(t)$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{v(t+h)-v(t)}{h} \\
&= \lim_{h\to 0} \frac{(-9.8(t+h))-(-9.8t)}{h} \\
&= \lim_{h\to 0} \frac{-9.8t-9.8h+9.8t}{h} \\
&= \lim_{h\to 0} \frac{-9.8h}{h} \\
&= \lim_{h\to 0} -9.8 \text{ assuming } h \neq 0 \\
&= -9.8.
\end{aligned}
$$

Which we calculated using difference quotients for fun. We can sketch this acceleration function, comparing it to the velocity function, since the acceleration at time t = the derivative of the velocity at time t= the slope of the tangent of the velocity function at time t:





The acceleration function is a constant, -9.8 m/s^2. Looks familiar? That's actually the acceleration induced by the earth's gravitational pull! So the acceleration when the ball is dropped, and when it hits the ground, is the same, -9.8m/s^2.

In summary:

Velocity is the instantaneous change in position and acceleration is the instantaneous change in velocity. Mathematically these are the derivatives of position and and velocity, respectively, with respect to time. If position is given by $s(t)$, then

$$
\begin{aligned}
\text{velocity } v(t) &= s'(t) \text{ or } \frac{ds}{dt} \\
\text{acceleration } a(t) &= v'(t) \text{ or } \frac{dv}{dt}.
\end{aligned}
$$