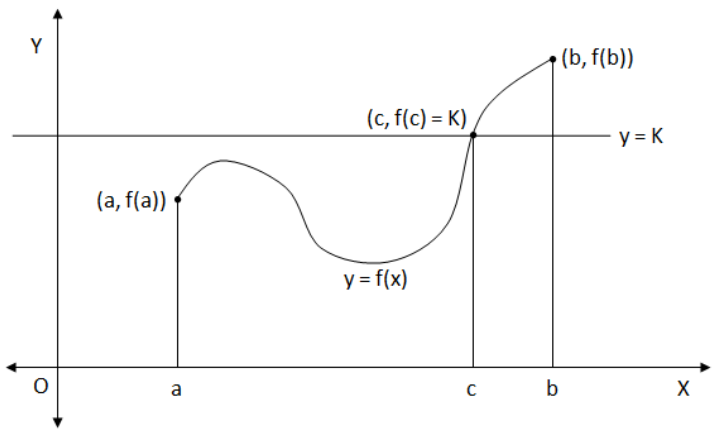
Intermediate Value Theorem

Theorem Suppose that the function f is continuous on the closed interval $a \leq x \leq b$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is, if K is any number between $f(a)$ and $f(b)$, then there exists at least one number c in $a < x < b$ such that $f(c) = K$.[1]



AN IMPORTANT APPLICATION of the Intermediate Value Theorem is the verification of the existence of solutions of equations written in the form

$$f(x) = 0.$$

EXAMPLE Show that $f(x) = x^3 - x - 2 = 0$ has a solution somewhere between $x = 1$ and $x = 2$.

SOLUTION The function $f(x)$ is continuous on $1 \leq x \leq 2$ because it is a polynomial and, therefore, continuous everywhere. Because $f(1) = -2 < 0$ and $f(2) = 4 > 0$, the IVT implies that every number between -2 and 4 is a value of $f(x)$ on $1 \leq x \leq 2$. In particular,

$$-2 = f(1) < 0 < f(2) = 4,$$

so the IVT for f implies that f attains the value 0 at some number c, $1 < c < 2$. That is, the IVT implies that there is at least one value c for c between 1 and 2 for which $f(c) = 0$ is true. Thus, by the IVT, $f(x) = 0$ has at least one solution in the interval $1 < x < 2$.

Notes:

(1) It's sufficient to say that $f(x)$ is continuous (say *why* it's continuous) and that $f(x)$ changes sign in the interval (*show* this by plugging in endpoints), and that therefore by the intermediate value theorem, $f(x) = 0$ has a solution on $1 < x < 2$.

(2) If you're not given an interval, you have to make one up!

[1] Picture from http://paramanands.blogspot.com/2011/06/continuous-functions-on-closed-interval-intermediate-value-theorem.html

Additional Problems

1. Show that $x^2 - 5 = 0$ has a solution on $[2, 3]$.
2. Show that $f(x) = x^3 + x + 1$ has a root in the interval $[-1, 0]$.
3. Show that $x^3 - 3x^2 + 1 = 0$ has a solution.
4. Show that $x^3 = 5$ has a solution on $[1, 2]$.
5. Show that $g(x) = x^4 + 2x - 1$ has at least one root.