Midterm 2 - In class review

Problems for the week leading up to the midterm. But remember, *everything* that we've covered since the last exam is fair game: lecture material, homework 7-9, quizzes 7&8, extra problems for optimization & related rates... *everything.*

Curve sketching with asymptotes

For each of the following,

(a) Find all x- and y- intercepts.

(b) Find all vertical and horizontal asymptotes (if any). Please be sure to justify your answers using limits.

(c) Find the intervals where f(x) is increasing or decreasing.

(d) Find and classify all local (aka relative) extrema.

(e) Find the intervals where f(x) is concave up or concave down.

(f) Find all points of inflection (if any).

(g) Sketch a graph of the function f(x) on the set of axis given below. Also please be sure to label your graph with all points found in parts (a)-(f).

1. $f(x) = x^2/(x^2-9)$

2. $g(x) = (x-1)/(x^2-2x-3)$

3. $h(x) = (x+1)^3/(x-1)^2$

Absolute maxima and minima

For the following, find the absolute maximum and minimum values of $f(x)$ on the given interval.

4. $f(x) = x^4 - 2x^2 + 3$; $[-2,3]$.

5. $f(x) = x - 2\cos(x)$; $[-\pi,\pi]$.

6. $f(x) = x^2 - 4x + 7$; $(0,\infty)$.

Maximum-minimum (aka optimization) problems

7. The measles pathogenesis function

$$f(t) = -t(t-21)(t+1)$$

can be used to model the development of the measles disease, where t is measured in days, and $f(t)$ represents the number of infected cells per milliliter of plasma. What is the peak infection time for the measles virus?

8. (a) A farmer with 2000 feet of fencing wants to enclose a rectangular area and then divide it into three pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the three pens?

(b) What are the dimensions of the pen that give the largest total area?

(c) & (d) Repeat parts (a) and (b), assuming that the farmer wants to divide the area into four pens instead.

9. (a) A box with square base and open top must have a volume of 4000 cm^3. Find the dimensions of the box that minimize the amount of material used.

(b) What is the largest possible volume a square box and open top that can be made from 1200 cm^2 of material?

Linearization

10. (a) Find the linearization of the function $f(x) = \sqrt[4]{x}$ at $x = 16$.

(b) Use the linearization above to estimate the value of $\sqrt[4]{16.32}$.

11. Find the linearization of $\sin(x)$ at $x = 0$. What does this say about values of $\sin(x)$ when x is close to zero (radians).

12. Use linearization of a suitable function to estimate each of the following:

(a) $(1.02)^3$

(b) $\sqrt[3]{0.99}$

(c) $\sqrt{83}$

Implicit differentiation

For each of the following functions, first solve for $\frac{dy}{dx}$, Then, find the equation of the tangent line to the function at a given point:

13. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$; $(0, 1/2)$.

14. $4\cos(x)\sin(y) = 1$;$(2\pi/3, 11\pi/6)$.

15. $x^{2/3} + y^{2/3} = 4$; $(3\sqrt{3}, 1)$.

Related rates problems

16. Sociologists have found that crime rates are influenced by temperature. In a midwestern town of 100,000 people, the crime rate has been approximated by:

$$C = \frac{1}{10}(T - 60)^2 + 100$$

where C is the number of crimes per month, and T is average temperature increase per month in degrees Fahrenheit. The average temperature for May is 76°F , and by the end of the month, the temperature is rising at a rate of 8°F per month. How fast is the crime rate rising at the end of May?

17. Two people leave from the same point at the same time. One runs northeast at 8 miles per hour, and the other bikes northwest at 12 miles per hour. How fast is the distance between the people changing after 15 minutes?

18. Recall that the volume and surface area of a sphere are given by $V = 4\pi r^3/3$ and $A = 4\pi r^2$.

(a) A spherical tumor is growing in such a way that its radius is increasing at a rate of 1 millimeter per week. Find the rate at which the volume is increasing at the time when the radius is 10mm.

(b) A spherical tumor is growing in such a way that its radius is increasing at a rate of 1 millimeter per week. Find the rate at which the surface area is increasing at the time when the radius is 10mm.

(c) A spherical tumor is growing in such a way that its surface area is increasing at a rate of 1 millimeter^2 per week. Find the rate at which the radius is increasing at the time when the radius is 10mm.

(d) A spherical tumor is growing in such a way that its volume is increasing at a rate of 1 millimeter^3 per week. Find the rate at which the radius is increasing at the time when the radius is 10mm.

Answers

1. (a) (0,0) is the only $x-$ or $y-$ intercept.

(b) Vertical asymptote at $x=-3$ since $\lim_{x\to-3} f(x)=\infty$. Vertical asymptote at $x=3$ since $\lim_{x\to 3} f(x)=\infty$. Horizontal asymptote at $y=1$ since $\lim_{x\to\pm\infty} f(x)=1$. No oblique asymptotes.

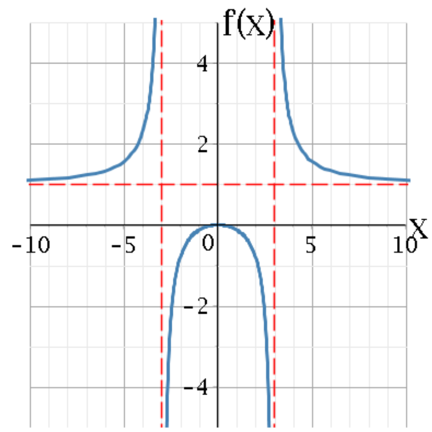
(c) $f(x)$ is increasing for $x<0$ and decreasing for $x>0$.

(d) Local maximum at $(0,0)$.

(e) $f(x)$ is concave up for $x<-3$ and $x>3$. $f(x)$ is concave down for $-3<x<3$.

(f) There are no points of inflection. $f''(x)$ does not exist at $x=\pm 3$ but these are asymptotes ($f(\pm 3)$ does not exist), not inflection points. That said, concavity does change at $x=\pm 3$.

(g) Sketch. **Note:** you're asked to label all the points. The labels are missing here (hard to do on a computer) but if they're missing from your sketch on an exam you lose points.



2. (a) $x-$intercept at (1,0). $y-$intercept at (0,1/3).

(b) Vertical asymptote at $x=-1$ since $\lim_{x\to-1} g(x)=\infty$. Vertical asymptote at $x=3$ since $\lim_{x\to 3} g(x)=\infty$. Horizontal asymptote at $y=0$ since $\lim_{x\to\pm\infty} g(x)=0$. No oblique asymptotes.

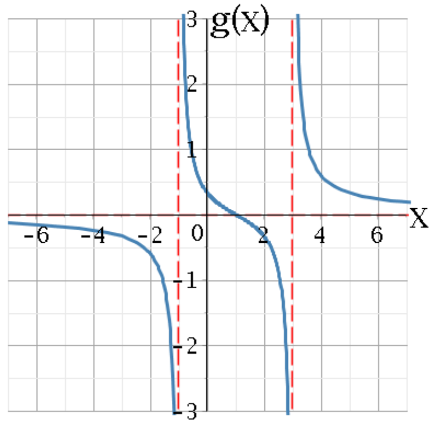
(c) $f(x)$ is decreasing for all x.

(d) No local extrema.

(e) $f(x)$ is concave down for $x<-1$ and $1<x<3$. $f(x)$ is concave down for $-1<x<1$ and $x>3$.

(f) Inflection point at $(1,0)$. $f''(x)$ does not exist at $x=-1$ or 3 but these are asymptotes ($f(-1)$, $f(3)$ do not exist), not inflection points. That said, concavity does change at $x=-1$ and $x=3$.

(g) Sketch. **Note:** you're asked to label all the points. The labels are missing here (hard to do on a computer) but if they're missing from your sketch on an exam you lose points.



3. (a) $x-$intercept at (-1,0). $y-$intercept at (0,1).

(b) Vertical asymptote at $x = 1$ since $\lim_{x\to 1} h(x) = \infty$. No horizontal asymptotes. Oblique asymptote at $y = x+5$ since $h(x) = x+5+\frac{12x-4}{(x-1)^2}$ and $\lim_{x\to\infty}\frac{12x-4}{(x-1)^2} = 0$.

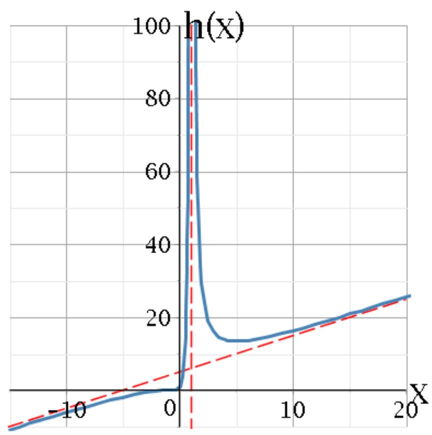
(c) $f(x)$ is increasing for $x < 1$ and $x > 5$. $f(x)$ is decreasing for $1 < x < 5$.

(d) Local minimum at $(5, 27/2)$.

(e) $f(x)$ is concave down for $x < -1$. $f(x)$ is concave up for $x > -1$.

(f) Inflection point at $(-1,0)$. $f''(x)$ does not exist at $x = 1$ BUT that's the asymptote ($f(1)$ does not exist), not and inflection point. *Note that concavity does not change at the asymptote.*

(g) Sketch. **Note:** you're asked to label all the points. The labels are missing here (hard to do on a computer) but if they're missing from your sketch on an exam you lose points.



4. Absolute minimum value is 2, attained at $x = -1$ and 1 (i.e. $f(\pm 1) = 2$). Absolute maximum value is 66, attained at $x = 3$ (i.e. $f(3) = 66$).

5. Absolute minimum value is $-\pi/6 - \sqrt{3}$, attained at $x = -\pi/6$ (i.e. $f(-\pi/6) = -\pi/6 - \sqrt{3}$). Absolute maximum value is $\pi + 2$, attained at $x = \pi$ (i.e. $f(\pi) = \pi + 2$).

6. Absolute minimum value is 3, attained at $x = 2$ (i.e. $f(2) = 3$. No absolute maximum value, not even at the endpoints, since it's an open interval.

7. $20/3+\sqrt{463}/3$ days, that is, approximately 13.84 days.

8. DRAW THIS ONE FOR SURE. Formulas just pop out of the sketch!

(a) If x is the length of the side parallel to fencing to split up the three pens, $x=250$ m, and $y=500$ m (length of the side perpendicular to the dividing fencing).

(b) The total area is $A=125000$ m^2.

(c) If x is the length of the side parallel to fencing to split up the four pens, $x=200$ m, and $y=500$ m (length of the side perpendicular to the dividing fencing).

(d) The total area is $A=100000$ m^2.

9. (a) Box dimensions 20 cm $\times$ 20 cm $\times$ 10 cm that minimize surface area for volume $V=4000$ cm^3.
(b) Box dimensions 20 cm $\times$ 20 cm $\times$ 10 cm that maximize volume given surface area $S=1200$ cm^2; that maximum volume is $V=4000$ cm^3.

10. (a) $L(x)=x/32+3/2$. (b) $\sqrt[4]{16.32}\approx 2.01$.

11. $L(x)=x$. Near $x=0$, $\sin(x)\approx x$.

12. (a) Use linearization of $f(x)=x^3$ at $x=1$. $L(x)=3x-2$. Therefore, $(1.02)^3=f(1.02)\approx L(1.02)=1.06$. That is, $(1.02)^3\approx 1.06$.

(b) Use linearization of $f(x)=\sqrt[3]{x}$ at $x=1$. $L(x)=x/3+2/3$. Therefore, $\sqrt[3]{0.99}=f(0.99)\approx L(0.99)=299/300$. That is, $\sqrt[3]{0.99}\approx 299/300$.

(c) Use linearization of $f(x)=\sqrt{x}$ at $x=9$. $L(x)=x/6+3/2$. Therefore, $\sqrt{83}=f(83)\approx L(83)=46/3$. That is, $\sqrt{83}\approx 46/3$.

13. $\frac{dy}{dx}=-\frac{x(4y^2+2x^2-1)}{y(4x^2+8y^2-1)}$; $y=1/2$.

14. $\frac{dy}{dx}=\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}$; $y=x+\frac{7\pi}{6}$.

15. $\frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}$; $y=-x/\sqrt{3}+2$.

16. Crime is rising at the rate $128/5\approx 25.6$ crimes per month.

17. 15 minutes later the people are moving apart at the rate $4\sqrt{13}$ miles/hour.

18. (a) Volume is growing at the rate 400π mm^3/week (b) Surface area is growing at the rate 80π mm^3/week (c) Radius is growing at the rate $1/80\pi$ mm/week. (d) Radius is growing at the rate $1/400\pi$ mm/week.