Maximum-minimum practice problems[1]
SOLUTIONS

1. For infants less than 9 months old, the relationship between rate of growth R (in lbs/month) and the present weight (W) is approximated by

$$R(W) = cW(21 - W)$$

for some constant c. At what weight is the growth rate the largest?

Solution:

We want to the weight W where the growth rate R is MAXIMIZED.

To get this extremum: first find critical points of $R(W)$. Need derivative $R'(W)$. Re-write $R(W) = 21cW - cW^2$. Then

$$R'(W) = 21c - 2cW$$

(c is constant). Derivative exists everywhere so critical points only where $R'(W) = 0$. Solving,

$$\begin{aligned} R'(W) &= 0 \\ 21c - 2cW &= 0 \\ W &= 21/2. \end{aligned}$$

The critical weight is $W = 21/2 = 10.5$ lbs (independent of the constant c!). To verify whether the growth rate is maximized, use the second derivative test.

$$\begin{aligned} R''(W) &= -2c \\ R''(10.5) &= -2c < 0. \end{aligned}$$

Therefore, according to this model, infants weighing 10.5 lbs are growing fastest.

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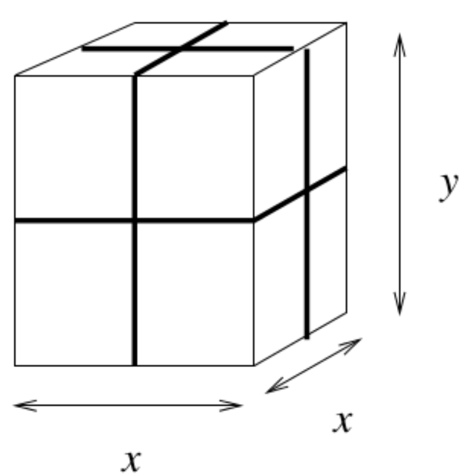
2. *Wrapping a rectangular box* A box with square base and arbitrary height has string tied around each of its perimeter. The total length of string so used is 10 inches. Find the dimensions of the box with largest surface area (how much wrapping paper do we need?)



[1] Problems 1,2,4 from Leah Keshet's *Differential Calculus*; Problem 3 from Bodine, Lenhart, Gross' *Mathematics for Life Sciences*.

Solution:

This is an optimization problem with a constraint: the length of string is 10 inches.

The surface area of the box is given by

$$S = 4(xy) + 2x^2$$

since there are two faces (top and bottom) which are squares (area x^2) and four rectangular faces with area xy.

We are constrained by the constant length of string $L = 10$ inches. It is given by 3 perimeters,

$$\begin{aligned} L &= 2(x+x) + 2(x+y) + 2x + y) = 8x + 4y \\ 10 &= 8x + 4y. \end{aligned}$$

We use this constraint to reduce the number of variables. Solving for y, $y = 5/2 - 2x$. Then the surface area becomes

$$\begin{aligned} S(x) &= 4x(\frac{5}{2} - 2x) + 4(\frac{5}{2} - 2x) \\ &= 10x - 6x^2. \end{aligned}$$

We want the length of the side x so that this surface area is maximized.

To get this extremum: first find critical points of $S(x)$. Need derivative $S'(x)$.

$$S'(x) = 10 - 12x.$$

Derivative exists everywhere so only critical points where $S'(x) = 0$. Solving,

$$\begin{aligned} S'(x) &= 0 \\ 10 - 12x &= 0 \\ x &= \frac{5}{6}. \end{aligned}$$

Critical point is $x = 5/6$ inch. To verify whether $S(5/6)$ actually a maximum, use the second derivative test.

$$\begin{aligned} S''(x) &= -12 \\ S''\left(\frac{5}{6}\right) &= -12 < 0 \end{aligned}$$

Therefore when the length of the side $x = 5/6$ inch, the surface area of the box is maximized. The $y-$value there is $y = 5/2 - 2(5/6) = 5/6$.

Thus the dimensions of the box with the maximized surface area are $5/6" \times 5/6" \times 5/6"$.

We're not asked for it but: $S(5/6) = 10(5/6) - 6(5/6)^2 = 25/6$ square inches is the max surface area. That's how much wrapping paper you need!

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3. In a simple model of territory, a single animal defends a circular area of radius r. Assume

1. Energy L spent looking for food and defending the territory is directly proportional to the *area* of the region.

2. Energy G gained is directly proportional to the radius r.

Suppose $L = 3000$ calories and $G = 3500$ calories when $r = 1$.

(a) Write down an equation for net energy $E = G - L$ as a function of r.

(b) Find the territorial size (defined by its radius r) that will result in maximum benefit to the animal, that is, maximize net energy $G - L$.

Solution:

(a) Energy $L = k_1 A = k_1 \pi r^2$ (k_1 is the proportionality constant). We can determine k_1 as follows: When $r = 1$, $L = 3000$, so

$$\begin{aligned} 3000 &= k_1 \pi (1)^2 \\ k_1 &= \frac{3000}{\pi}. \end{aligned}$$

Therefore, $L = \left(\frac{3000}{\pi}\right) \pi r^2 = 3000r^2$.

Energy $G = k_2 r$ (k_2 is the proportionality constant). We can determine k_2 as follows: When $r = 1$, $G = 3500$, so

$$\begin{aligned} 3500 &= k_2(1) \\ k_2 &= 3500. \end{aligned}$$

Therefore, $G = 3500r$.

Net energy is given by $E = G - L$. We now know G and L in terms of r. Therefore E in terms of r is

$$\begin{aligned} E &= G - L \\ &= 3500r - 3000r^2 \\ E &= 500r(7 - 6r). \end{aligned}$$

The net energy is given by $E(r) = 500r(7 - 6r)$.

(b) We want the territorial size in terms of the radius r that will maximize the net energy $E = G - L$. To get this extremum: first find critical points of $E(r)$. Need derivative $E'(r)$.

$$E'(r) = 3500 - 6000r.$$

Derivative exists everywhere so only critical points where $S'(x) = 0$. Solving,

$$\begin{aligned} E'(r) &= 0 \\ 3500 - 6000r &= 0 \\ r &= \frac{7}{12}. \end{aligned}$$

Critical radius is $r = 12/7$. To verify whether $E(12/7)$ actually a maximum, use the second derivative test.

$$\begin{aligned} E''(x) &= -6000 \\ E''\left(\frac{7}{12}\right) &= -6000 < 0 \end{aligned}$$

Therefore when the radius $r = 7/12$, the net energy gained from defending the territory is maximized.

The area of the territory A is $A = \pi(7/12)^2 = 49\pi/144 \approx 1.1$.

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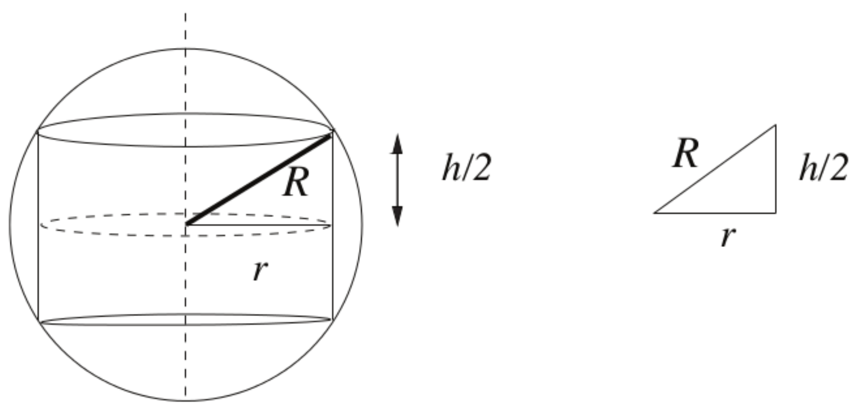
4. *Fitting a cylinder inside a sphere* Find the cylinder of maximal volume that would fit inside a sphere of radius R.

Solution:

Tricky one! As recommended by our strategies, let's draw a picture to get a handle on this:



The radius R is a fixed constant. r, h are the dimensions of the cylinder, to be determined.

The volume to maximize is $V = \pi r^2 h$.

We need to include the constraint - that the cylinder is inscribed in the sphere of radius R - which will reduce the number of variables. Check out the picture above: we can use the Pythagorean Theorem to determine a relationship between r and h:

$$\begin{aligned} R^2 &= r^2 + (h/2)^2 \\ r^2 &= R^2 - (h/2)^2. \end{aligned}$$

(we keep r^2 since r^2 is in the volume equation V). Then the volume to be maximized is given by

$$\begin{aligned} V &= \pi h\left(R^2 - \frac{h^2}{4}\right) \\ &= \pi h R^2 - \frac{\pi h^3}{4}. \end{aligned}$$

To get this extremum: first find critical points of $V(h)$. Need derivative $V'(h)$.

$$V'(h) = \pi R^2 - \frac{3\pi h^2}{4}$$

Derivative exists everywhere so only critical points where $V'(h) = 0$. Solving,

$$\begin{aligned} V'(h) &= 0 \\ \pi R^2 - \frac{3\pi h^2}{4} &= 0 \\ h &= \frac{2R}{\sqrt{3}} \end{aligned}$$

(we neglect the negative root $h = -2R/\sqrt{3}$ since it's a height). Critical cylinder height is $r = 2R/\sqrt{3}$. To verify whether $V\left(2R/\sqrt{3}\right)$ actually a maximum, use the second derivative test.

$$\begin{aligned} V''(h) &= -\frac{3\pi h}{2} \\ V''\left(\frac{2R}{\sqrt{3}}\right) &= -\frac{3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right) < 0. \end{aligned}$$

Therefore the cylinder of height $h = \left(2R/\sqrt{3}\right)$ has the max possible volume in a sphere of radius R.

To get the volume, just plug in: $V(2R/\sqrt{3}) = \pi(2R/\sqrt{3})R^2 - \pi\left(2R/\sqrt{3}\right)^3/4$, or

$$V = \frac{4\pi R^3}{3\sqrt{3}}.$$