Practice final exam

1. Evaluate each of the following limits (if it exists). If the limit exists, be sure to justify your answer; if the limit does not exist, be sure to explain why.

$$\text{(a)} \lim_{x\to 2}\frac{x^2-4x+4}{x^2-5x+6}, \qquad \text{(b)} \lim_{x\to 3}\frac{x^2-4x+4}{x^2-5x+6}, \qquad \text{(c)} \lim_{x\to \infty}\frac{x^2-4x+4}{x^2-5x+6}.$$

2. Differentiate each of the following functions (using whichever method you like), and evaluate at the given value of x, if given. Please be sure to show all work and simplify your final answer.

(a) $f(x) = (3x-1)^2$, $x = 3$ (b) $g(x) = e^{x\ln(x)}$, $x = 2$
(c) $h(x) = \sin^2(\ln(\sqrt{x}))$, $x = 1$ (d) $w(x) = \frac{x}{1-\sin^2(x)}$, $x = 0$

3. Consider the function

$$f(x) = \frac{4}{1+e^{1-x}}$$

(a) Find all x- and y- intercepts.
(b) Find any and all asymptote. Justify your answers using limits.
(c) Find the intervals where the function is increasing or decreasing.
(d) Find and classify all local extrema, if any.
(e) Find the intervals where the function is concave up or concave down.
(f) Find all points of inflection, if any.
(g) Sketch a graph of the function, labeling all points.

4. (a) Find the derivative dy/dx of the curve $e^y\cos(x) = 1+\sin(xy)$.

(b) Use the result from part (a) above to find the equation of the line tangent to the curve $e^y\cos(x) = 1+\sin(xy)$ at the point $(0,0)$.

5. Consider the function

$$f(x) = \begin{cases} -\frac{1}{x}, & x < 0 \\ \frac{1}{x}, & 0 < x < 1 \\ 1, & 1 \le x < 3 \\ 0, & 3 \le x \le 4 \\ 1, & x > 4. \end{cases}$$

(a) Sketch the function $f(x)$.

(b) Find and classify all points at which f is discontinuous.

(c) Find all points where f is not differentiable.

6. A rectangular piece of cardboard with dimension 12 cm by 24 cm is to be made into an open box (i.e., no lid) by cutting out squares from the corners and then turning up the sides. Find the size of the squares that should be cut out if the volume of the box is to be a maximum.

7. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

8. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420 cells.

(a) Write an expression in the form $b(t) = b_0 e^{kt}$ for the number of bacteria b after t hours, i.e. with numerical values for b_0 and k.

(b) Write down the differential equation $b(t)$ satisfies.

(c) What is the bacterial population doubling time?

(d) Find the number of bacteria after 3 hours.

(e) When will the population reach 20 000 cells?

Note: you may leave your answers as expressions with "e" or "ln" where necessary

9. Suppose $S'(t)$ is the rate of sale of Valentine's day candy, where $S(t)$ is the accumulated pounds of Valentine's day candy sold and t is the number of days since the beginning of February, i.e., $t = 1$ gives February first, etc.

(a) Give the significance of

$$\int_0^{14} S'(t)\, dt.$$

(b) Will $S'(15)$ be positive, negative, or zero? Explain.

(c) Will $S'(28)$ be positive, negative, or zero? Explain.

10. Suppose the acceleration function $a(t)$ of a particle moving in a straight line is given by $a(t) = \cos(t) + \sqrt{t}$, measured in meters per second^2.

(a) What is the initial acceleration?

(b) What is the particle's velocity at time $t = 9$, if it was initially at rest ($v(0) = 0$ m/s?)? Check your answer by using this velocity function to recover an acceleration function and compare with $a(t) = \cos(t) + \sqrt{t}$.

(c) What is the particle's displacement at time t if its initial position is $s(0) = 10$ m?

11. (a) Sketch the function $f(x) = x^2 - 3x$ on the interval $[0, 3]$.

(b) Using a Riemann sum with 6 sub-intervals, find an approximation of the integral $\int_0^3 (x^2 - 3x)\, dx$.

(c) What sign is your answer? Interpret using signed areas.

12. Find the area enclosed by $y = x^2 + 6$ and $y = 8x - x^2$.

13. Integrate

(a) $\displaystyle\int_0^2 \left(x^e + e^{x/2}\right) dx$ (b) $\int_1^2 \frac{v^3+3v^6}{v^4}\, dv$ (c) $\displaystyle\int (\cos(2x) - \sin(x))\, dx$

(d) $\displaystyle\int_1^8 \sqrt[3]{x}\, dx$ (e) $\int_1^{18} \sqrt{\frac{3}{z}}\, dz$ (f) $\displaystyle\int (1-t)\left(2+t^2\right) dt$