Practice final exam

1. Evaluate each of the following limits (if it exists). If the limit exists, be sure to justify your answer; if the limit does not exist, be sure to explain why.

$$\text{(a)}\ \lim_{x\to 2}\frac{x^2-4x+4}{x^2-5x+6}, \qquad \text{(b)}\ \lim_{x\to 3}\frac{x^2-4x+4}{x^2-5x+6}, \qquad \text{(c)}\ \lim_{x\to \infty}\frac{x^2-4x+4}{x^2-5x+6}.$$

Solution: (a)

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4x+4}{x^2-5x+6} &= \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x-3)}\\ &= \lim_{x\to 2}\frac{(x-2)}{(x-3)}\\ &= \frac{(2-2)}{(2-3)}\\ &= 0.\end{aligned}$$

Thus,

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^2-5x+6} = 0.$$

(b)

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-4x+4}{x^2-5x+6} &= \lim_{x\to 3}\frac{(x-2)^2}{(x-2)(x-3)}\\ &= \lim_{x\to 3}\frac{(x-2)}{(x-3)}\\ &= \text{does not exist.}\end{aligned}$$

The limit $\lim_{x\to 3}\left(x^2-4x+4\right)/\left(x^2-5x+6\right)$ does not exist (undefined) because as $x\to 3$, $(x-2)/(x-3)\to\infty$, that is, the denominator is zero at $x=3$.

(c)

$$\begin{aligned}\lim_{x\to \infty}\frac{x^2-4x+4}{x^2-5x+6} &= \lim_{x\to \infty}\left(\frac{x^2-4x+4}{x^2-5x+6}\right)\left(\frac{1/x^2}{1/x^2}\right)\\ &= \lim_{x\to \infty}\left(\frac{1-4/x+4/x^2}{1-5/x+6/x^2}\right)\left(\frac{1/x^2}{1/x^2}\right) Thus,\\ &= 1.\end{aligned}$$

Thus,

$$\lim_{x\to \infty}\frac{x^2-4x+4}{x^2-5x+6} = 1.$$

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2. Differentiate each of the following functions (using whichever method you like), and evaluate at the given value of x, if given. Please be sure to show all work and simplify your final answer.

(a) $f(x) = (3x-1)^2$, $x = 3$ (b) $g(x) = e^{x\ln(x)}$, $x = 2$
(c) $h(x) = \sin^2(\ln(\sqrt{x}))$, $x = 1$ (d) $w(x) = \frac{x}{1-\sin^2(x)}$, $x = 0$

Solution: (a) First find $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left((3x-1)^2\right) \\ &= \frac{d}{dx}\left(9x^2-6x+1\right) \\ &= 18x-6. \end{aligned}$$

Then evaluate $f'(3)$:

$$\begin{aligned} f'(3) &= 18(3)-6 \\ &= 48. \end{aligned}$$

Thus $f'(3) = 48.$
(b) First find $g'(x)$:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{x\ln(x)}\right) \\ &= e^{x\ln(x)}\frac{d}{dx}(x\ln(x)) \\ &= e^{x\ln(x)}\left(\ln(x)+x\left(\frac{1}{x}\right)\right) \\ &= e^{x\ln(x)}\left(\ln(x)+1\right). \end{aligned}$$

But $e^{x\ln(x)} = e^{\ln(x^x)} = x^x$, so

$$g'(x) = x^x\left(\ln(x)+1\right).$$

Then evaluate $g'(2)$:

$$\begin{aligned} g'(2) &= (2)^{(2)}\left(\ln(2)+1\right) \\ &= 4\left(\ln(2)+1\right). \end{aligned}$$

Thus $g'(2) = 4\left(\ln(2)+1\right).$
(c) First find $h'(x)$:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\sin^2(\ln(\sqrt{x}))\right) \\ &= 2\sin(\ln(\sqrt{x}))\cos(\ln(\sqrt{x}))\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{1}{x}\sin(\ln(\sqrt{x}))\cos(\ln(\sqrt{x})) \end{aligned}$$

Then evaluate $h'(1)$:

$$\begin{aligned} h'(1) &= \frac{1}{(1)}\sin(\ln(\sqrt{(1)}))\cos(\ln(\sqrt{(1)})) \\ &= \sin(\ln(1))\cos(\ln(1)) \\ &= \sin(0)\cos(0) \\ &= 0 \end{aligned}$$

Thus $h'(1) = 0$.
Note: $\ln(\sqrt{x}) = \ln(x)/2$. This may make the derivative easier! $h(x) = \sin^2(\ln(x)/2)$, so

$$h'(x) = 2\sin(\ln(x)/2)\cos(\ln(x)/2)\,(1/2x) = ...$$

(d) $w(x) = \frac{x}{1-\sin^2(x)}, x = 0$ First find $w'(x)$,

$$\begin{aligned} w'(x) &= \frac{\left(1-\sin^2(x)\right)\frac{d}{dx}(x) - (x)\frac{d}{dx}\left(1-\sin^2(x)\right)}{\left(1-\sin^2(x)\right)^2} \\ &= \frac{\left(1-\sin^2(x)\right)(1) - (x)\left(-2\sin(x)\cos(x)\right)}{\left(1-\sin^2(x)\right)^2} \\ &= \frac{1-\sin^2(x) + 2x\sin(x)\cos(x)}{\left(1-\sin^2(x)\right)^2}. \end{aligned}$$

Then evaluate $w'(0)$:

$$\begin{aligned} w'(0) &= \frac{1-\sin^2(0) + 2(0)\sin(0)\cos(0)}{\left(1-\sin^2(0)\right)^2} \\ &= 1. \end{aligned}$$

Thus $w'(0) = 0$.

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3. Consider the function

$$f(x) = \frac{4}{1+e^{1-x}}$$

(a) Find all x- and y- intercepts.
(b) Find any and all asymptote. Justify your answers using limits.
(c) Find the intervals where the function is increasing or decreasing.
(d) Find and classify all local extrema, if any.
(e) Find the intervals where the function is concave up or concave down.
(f) Find all points of inflection, if any.
(g) Sketch a graph of the function, labeling all points.

Solution: (a) Intercepts: There is a $y-$intercept at $y = f(0) = 4/(1-e)$.
If there were an $x-$intercept it would satisfy $f(x) = 0 \Rightarrow 4/\left(1+e^{1-x}\right) = 0 \Rightarrow 4 = 0$, since $1+e^{1-x} > 0$ for all x. But $4 = 0$ is nonsense! There are no $x-$intercepts.
(b) Asymptotes: as $1+e^{1-x} \neq 0$ for all x, there are no vertical asymptotes. There are two horizontal asymptotes, at $x = 4$ and at $x = 0$, since $\lim_{x\to\infty} 4/\left(1+e^{1-x}\right) = 4$ and $\lim_{x\to-\infty} 4/\left(1+e^{1-x}\right) = 0$.
(c) First find the first derivative,

$$\begin{aligned} f'(x) &= \frac{\left(1+e^{1-x}\right)\frac{d}{dx}(4) - (4)\frac{d}{dx}\left(1+e^{1-x}\right)}{(1+e^{1-x})^2} \\ &= \frac{4e^{1-x}}{(1+e^{1-x})^2}. \end{aligned}$$

There are no solutions of $f'(x) = 0$, and $f'(x)$ exists everywhere, so there are no critical points. $f'(x) > 0$ for all values of x so $f(x)$ is increasing for all x.
Thus, $f(x)$ is increasing on $-\infty < x < \infty$.
(d) No local points so no local extrema!
(e) First find the second derivative,

$$\begin{aligned} f''(x) &= \frac{\left(1+e^{1-x}\right)^2\frac{d}{dx}\left(4e^{1-x}\right) - \left(4e^{1-x}\right)\frac{d}{dx}\left(\left(1+e^{1-x}\right)^2\right)}{(1+e^{1-x})^4} \\ &= \frac{\left(1+e^{1-x}\right)^2\left(-4e^{1-x}\right) - \left(4e^{1-x}\right)\left(2\left(1+e^{1-x}\right)\left(-e^{1-x}\right)\right)}{(1+e^{1-x})^4} \\ &= \frac{4e^{1-x}\left(e^{1-x}-1\right)}{(1+e^{1-x})^3}. \end{aligned}$$

$f''(x)$ exists everywhere, but there is a potential inflection point where $f''(x) = 0$:

$$\begin{aligned} \frac{4e^{1-x}\left(e^{1-x}-1\right)}{(1+e^{1-x})^3} &= 0 \\ e^{1-x}-1 &= 0 \\ e^x &= e \\ x &= 1 \end{aligned}$$

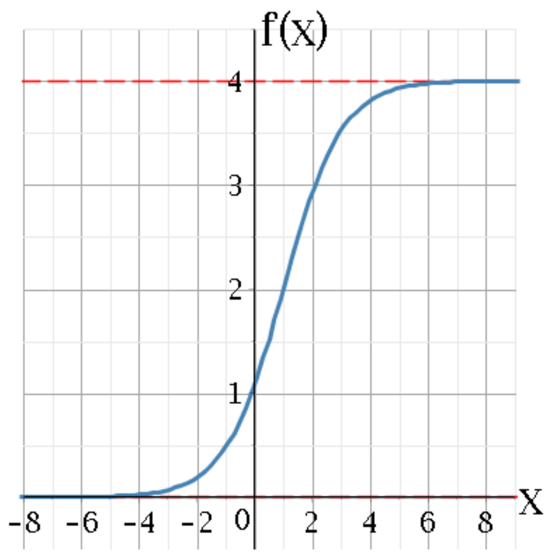
(we used the fact that $e^{1-x} \neq 0$ and $\left(1+e^{1-x}\right)^3 \neq 0$ for any value of x).
Check concavity:

	$x < 1$	$x > 1$
Test point x^*	0	2
$f''(x^*)$	$4e^1(e^1-1)/\left(1+e^1\right)^3 > 0$	$4e^{-1}(e^{-1}-1)/\left(1+e^{-1}\right)^3 < 0$
Behavior of $f(x)$	concave up.	concave down

Therefore, $f(x)$ is concave up for $x < 1$ and concave down for $x > 1$.

(f) There is an inflection point at $x = 1$ since concavity changes at that point.

(g) Sketch... Connect the dots.



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4. (a) Find the derivative dy/dx of the curve $e^y \cos(x) = 1 + \sin(xy)$.

(b) Use the result from part (a) above to find the equation of the line tangent to the curve $e^y \cos(x) = 1 + \sin(xy)$ at the point $(0,0)$.

Solution: (a) Need to employ implicit differentiation,

$$\begin{aligned}
\frac{d}{dx}\left(e^y \cos(x)\right) &= \frac{d}{dx}\left(1 + \sin(xy)\right) \\
e^y \cos(x)\frac{dy}{dx} - e^y \sin(x) &= \cos(xy)\left(y + \frac{dy}{dx}\right) \\
\Rightarrow \frac{dy}{dx} &= \frac{e^y \sin(x) + y\cos(xy)}{e^y \cos(x) - x\cos(xy)}.
\end{aligned}$$

(b) The slope m of the curve at $(0,0)$ is given by

$$\begin{aligned}
m &= \left.\frac{dy}{dx}\right|_{x=0,y=0} \\
&= \frac{e^{(0)} \sin(0) + (0)\cos(0)}{e^{(0)} \cos(0) - (0)\cos(0)} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}$$

Then we find the equation of the line using the point-slope fomula, with the point $(x_1, y_1) = (0,0)$:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 0(x - 0) \\ y &= 0. \end{aligned}$$

The equation of the line is $y = 0$.

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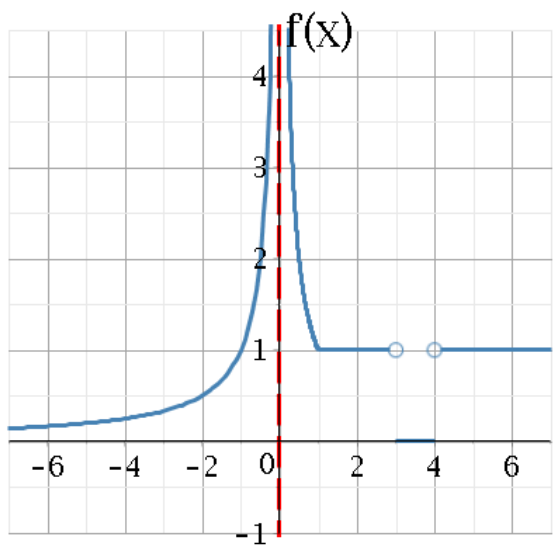
5. Consider the function

$$f(x) = \begin{cases} -\frac{1}{x}, & x < 0 \\ \frac{1}{x}, & 0 < x < 1 \\ 1, & 1 \leq x < 3 \\ 0, & 3 \leq x \leq 4 \\ 1, & x > 4. \end{cases}$$

(a) Sketch the function $f(x)$.
(b) Find and classify all points at which f is discontinuous.
(c) Find all points where f is not differentiable.
Solution: (a)



(b) $f(x)$ is not continuous at:

$x = 0$	infinite discontinuity
$x = 3$	jump discontinuity
$x = 4$	jump discontinuity

(c) $f(x)$ is not differentiable at $x = 0$, $x = 3$, or $x = 4$, because it is not continuous there. $f(x)$ is also not differentiable at $x = 1$ (corner).

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6. A rectangular piece of cardboard with dimension 12 cm by 24 cm is to be made into an open box (i.e., no lid) by cutting out squares from the corners and then turning up the sides. Find the size of the squares that should be cut out if the volume of the box is to be a maximum.

Solution: Definitely sketch this (hard in typewritten form).

Want to maximize the volume, $V = \text{width} \times \text{length} \times \text{height}$.

The height is given by x, the edge length of the squares being cut out. The width is given by $12 - 2x$, short side minus what's taken off with the two squares on either side. And the length is given by $24 - 2x$, long side minus what's taken off with the two squares on either side.

We therefore want to maximize $V = (x)(12-2x)(24-2x) = 4x^3 - 72x^2 + 288x$.

First take derivative,

$$V'(x) = 12x^2 - 144x + 288.$$

The critical points are the solutions of $V'(x) = 0$. Using the quadratic formula,

$$x = 6 + 2\sqrt{3} \text{ and } 6 - 2\sqrt{3}.$$

Now, $6 + 2\sqrt{3}$ is actually outside the interval of x: the width has to be positive, so $12 - 2x > 0 \Rightarrow x < 6$. So the only possible $x-$value is $6 - 2\sqrt{3}$.

Next we verify that $x = 6 - 2\sqrt{3}$ maximizes the volume $V(x)$. Use the second derivative test. $V''(x) = 24x - 144 = 24(x-6)$ so

$$\begin{aligned} V''(6-2\sqrt{3}) &= 24(6-2\sqrt{3}-6) \\ &= -24\left(2\sqrt{3}\right) \\ V''(6-2\sqrt{3}) &< 0. \end{aligned}$$

Since $V''(6-2\sqrt{3}) < 0$, $x = 6 - 2\sqrt{3}$ maximizes $V(x)$.

Finally we find the dimensions of the box.

$$\begin{aligned} &\text{height} = x = 6 - 2\sqrt{3} \\ &\text{width} = 12 - 2x = 4\sqrt{3} \\ &\text{length} = 24 - 2x = 12 + 4\sqrt{3}. \end{aligned}$$

Therefore the dimensions that maximize the volume of the box are $4\sqrt{3} \times 12 + 4\sqrt{3} \times 6 - 2\sqrt{3}$.

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7. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Solution: Definitely sketch this (hard in typewritten form). The area of the rectangle is $A(t) = w(t)\ell(t)$ where $w(t)$ is width at time t and $\ell(t)$ is the length at time t. We are given that

$$\frac{dw}{dt} = 3 \text{ cm/s} \ \text{ and } \frac{d\ell}{dt} = 8 \text{ cm/s}.$$

We are asked the rate of change of the area $A(t)$ when the width $w(t) = 10$ cm and the length $\ell(t) = 20$ cm.

The rate of change in the area is dA/dt:

$$\begin{aligned} \frac{dA}{dt} &= \frac{d}{dt}(w(t)\ell(t)) \\ &= w\frac{d\ell}{dt} + \ell\frac{dw.}{dt} \end{aligned}$$

So when the width $w(t) = 10$ cm and the length $\ell(t) = 20$ cm,

$$\begin{aligned} \frac{dA}{dt} &= (10 \text{ cm})(8 \text{ cm/s}) + (20 \text{ cm})(3 \text{ cm/s}) \\ &= 80 \text{ cm}^2/\text{s} + 60 \text{ cm}^2/\text{s} \\ &= 140 \text{ cm}^2/\text{s}. \end{aligned}$$

Therefore, the area of the rectangle is increasing at the rate $dA/dt = 140$ cm^2/s.

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8. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420 cells.

(a) Write an expression in the form $b(t) = b_0e^{kt}$ for the number of bacteria b after t hours, i.e. with numerical values for b_0 and k.

(b) Write down the differential equation $b(t)$ satisfies.

(c) What is the bacterial population doubling time?

(d) Find the number of bacteria after 3 hours.

(e) When will the population reach 20 000 cells?

Note: you may leave your answers as expressions with "e" or "ln" where necessary

Solution: (a) $b(t) = b_0e^{kt}$. We are given $b_0 = 100$ cells, so $b(t) = 100e^{kt}$. To find k use the fact that after an hour, the population has 420 cells – that is,

$$\begin{aligned} b(1) &= 420 \\ 100e^k &= 420 \\ \Rightarrow k &= \ln(21/5) = \ln(21) - \ln(5). \end{aligned}$$

Therefore, $b(t) = 100e^{\ln(21/5)t}$. We can simplify further, using properties of logarithms, to obtain $b(t) = 100(21/5)^t$, but this form is not useful for (b) or (c) (and anyway we weren't asked for it!).

(b) $b(t)$ satisfies the differential equation

$$b'(t) = \ln\left(\frac{21}{5}\right) b(t).$$

Note: The initial condition $b(0)$ is not asked for. However, if you were asked for the initial value problem that $b(t)$ satisfies, the answer would include the initial condition, i.e.

$$\begin{cases} b'(t) = \ln\left(\frac{21}{5}\right) b(t) \\ b(0) = 100 \text{ cells.} \end{cases}$$

(c) The bacterial population doubling time is

$$T = \frac{\ln(2)}{\ln(21/5)} \text{ hours} = \frac{\ln(2)}{\ln(21) - \ln(5)} \text{ hours.}$$

Notice the lack of decimal values. No calculators on the exam!
(d) After 3 hours, the number of bacteria is $b(3) = 100e^{3\ln(21/5)} = 100(21/5)^3$ cells.
(e) The population will reach 20 000 cells at time τ that satisfies $b(\tau) = 20000$, that is,

$$\begin{aligned} 100e^{\ln(21/5)\tau} &= 20000 \\ e^{\ln(21/5)\tau} &= 200 \\ \tau &= \frac{\ln(200)}{\ln(21) - \ln(5)} \text{ hours.} \end{aligned}$$

Therefore the population will reach 20 000 cells after $\tau = \ln(200)/(\ln(21) - \ln(5))$ hours.
When will the population reach 20 000 cells?

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9. Suppose $S'(t)$ is the rate of sale of Valentine's day candy, where $S(t)$ is the accumulated pounds of Valentine's day candy sold and t is the number of days since the beginning of February, i.e., $t = 1$ gives February first, etc.
(a) Give the significance of

$$\int_0^{14} S'(t)\,dt.$$

(b) Will $S'(15)$ be positive, negative, or zero? Explain.
(c) Will $S'(28)$ be positive, negative, or zero? Explain.

Solution: (a) $\int_0^{14} S'(t)\,dt$ represents the accumulated pounds of candy sold from the start of February up to February 14 (Valentine's day).
(b) Expect $S'(15)$ to be positive since $S(t)$ is the accumulated pounds of candy sold. No negative candy! But sales are slowing down so we would expect, for example, $S'(15) < S'(28)$.
(c) Expect $S'(28)$ to be zero – no more Valentine's day candy is being sold by the end of the month.

10. Suppose the acceleration function $a(t)$ of a particle moving in a straight line is given by $a(t) = \cos(t) + \sqrt{t}$, measured in meters per second^2.

(a) What is the initial acceleration?

(b) What is the particle's velocity at time $t = 9$, if it was initially at rest ($v(0) = 0$ m/s?)? Check your answer by using this velocity function to recover an acceleration function and compare with $a(t) = \cos(t) + \sqrt{t}$.

(c) What is the particle's displacement if its initial position is $s(0) = 10$ m?

Solution: (a) The initial acceleration is given by $a(0) = \cos(0) + \sqrt{(0)} = 1$.That is, the initial acceleration is $a(0) = 1$.

(b) First we find the velocity function.

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int \left(\cos(t) + \sqrt{t}\right)\,dt \\ &= \sin(t) + \frac{2t^{3/2}}{3} + C. \end{aligned}$$

To determine C, use $v(0) = 0$:

$$\begin{aligned} 0 &= \sin(0) + \frac{2(0)^{3/2}}{3} + C \\ \Rightarrow C &= 0. \end{aligned}$$

Therefore, $v(t) = \sin(t) + 2t^{3/2}/3$.

Then the velocity at time $t = 9$ is $v(9)$:

$$\begin{aligned} v(9) &= \sin(9) + \frac{2(9)^{3/2}}{3} \\ &= \sin(9) + 18. \end{aligned}$$

Thus at $t = 9$ the velocity is $v(9) = \sin(9) + 18$.

(c) Now find the displacement function:

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int \left(\sin(t) + \frac{2t^{3/2}}{3}\right)\,dt \\ &= -\cos(t) + \frac{4t^{5/2}}{15} + C. \end{aligned}$$

To determine C, use $s(0) = 10$.

$$\begin{aligned} 10 &= -\cos(0) + \frac{4(0)^{5/2}}{15} + C \\ \Rightarrow C &= 11. \end{aligned}$$

Therefore, the displacement at time t is $s(t) = -\cos(t) + 4t^{5/2}/15 + 11$.

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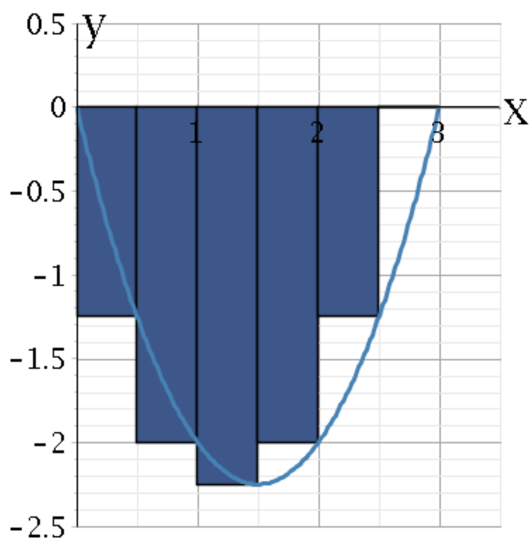
11. (a) Sketch the function $f(x) = x^2 - 3x$ on the interval $[0,3]$.
(b) Using a Riemann sum with 6 sub-intervals, find an approximation of the integral $\int_0^3 (x^2 - 3x)\,dx$.
(c) What sign is your answer? Interpret using signed areas.

Solution: (a) We sketch the function – also included here are the "Riemann sum" rectangles (not asked for in the problem, but may help you understand (b)).



(b) The integral can be approximated as

$$\int_0^3 \left(x^2 - 3x\right)\,dx \approx \sum_{i=1}^{6} f(x_i)\Delta x,$$

where $\Delta x = (3-0)/6 = 1/2$ (the denominator is 6 because there are 6 sub-intervals), $x_i = 0 + i\Delta x =$

$i/2$, and $f(x) = x^2 - 3x$. Therefore

$$\begin{aligned}
\int_0^3 \left(x^2 - 3x\right)\, dx &\approx \sum_{i=1}^{6} \frac{1}{2} \left(x_i^2 - 3x_i\right) \\
&= \frac{1}{2} \sum_{i=1}^{6} \left(\left(\frac{i}{2}\right)^2 - 3\left(\frac{i}{2}\right) \right) \\
&= \frac{1}{2} \sum_{i=1}^{6} \left(\frac{i^2}{4} - \frac{3i}{2} \right) \\
&= \frac{1}{8} \sum_{i=1}^{6} \left(i^2 - 6i\right)
\end{aligned}$$

is the Riemann sum that approximates the integral with 6 sub-intervals.

Next we compute the approximation,

$$\int_0^3 \left(x^2 - 3x\right)\, dx \approx -\frac{35}{8}.$$

Thus, $\int_0^3 \left(x^2 - 3x\right)\, dx \approx \frac{1}{8} \sum_{i=1}^{6} \left(i^2 - 6i\right) = -35/8$.

(c) The sign of the answer is negative. We interpret this to mean that the area under the $x-$axis is greater than the area above the $x-$axis (since there's nothing above the $x-$axis this much is clear!).

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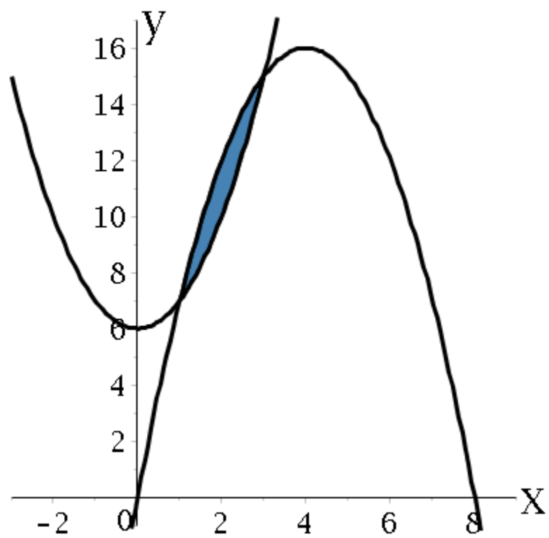
12. Find the area enclosed by $y = x^2 + 6$ and $y = 8x - x^2$.

Solution: First find the points where the curves intersect by solving $x^2 + 6 = 8x - x^2$:

$$\begin{aligned}
x^2 + 6 &= 8x - x^2 \\
2x^2 - 8x + 6 &= 0 \\
2(x-1)(x-3) &= 0,
\end{aligned}$$

which has solutions $x = 1$ and $x = 3$. Therefore our interval is $[1, 3]$; these will be the limits of integration.

Next, we sketch:



Note that $y = 8x - x^2$ is the upper graph in the interval $[1,3]$, that is, that $8x - x^2 \geq x^2 + 6$ for $1 \leq x \leq 3$.

Now we can compute the area (shaded region in the sketch above).

$$
\begin{aligned}
\text{Area } &= \int_1^3 \left(\left(8x - x^2\right) - \left(x^2 + 6\right)\right)\, dx \\
&= \int_1^3 \left(-2x^2 + 8x - 6\right)\, dx \\
&= \left(-\frac{2x^3}{3} + 4x^2 - 6x\right)\Bigg|_0^1 \\
&= \left(-\frac{2(1)^3}{3} + 4(1)^2 - 6(1)\right) - \left(-\frac{2(0)^3}{3} + 4(0)^2 - 6(0)\right) \\
&= \frac{8}{3}.
\end{aligned}
$$

Therefore the area enclosed by $y = 8x - x^2$ and $y = x^2 + 6$ is 8/3.

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13. Integrate

(a) $\displaystyle\int_0^2 \left(x^e + e^{x/2}\right)\, dx$ (b) $\int_1^2 \frac{v^3+3v^6}{v^4}\, dv$ (c) $\displaystyle\int (\cos(2x) - \sin(x))\, dx$

(d) $\displaystyle\int_1^8 \sqrt[3]{x}\, dx$ (e) $\int_1^{18} \sqrt{\frac{3}{z}}\, dz$ (f) $\displaystyle\int (1-t)\left(2+t^2\right)\, dt$

Solution:

(a)

$$\begin{aligned}
\int_0^2 \left(x^e + e^{x/2}\right)\, dx &= \left.\frac{x^{e+1}}{e+1} + 2e^{x/2}\right|_0^2 \\
&= \left(\frac{(2)^{e+1}}{e+1} + 2e^{2/2}\right) - \left(\frac{(0)^{e+1}}{e+1} + 2e^{(0)/2}\right) \\
\int_0^2 \left(x^e + e^{x/2}\right)\, dx &= \frac{2^{e+1}}{e+1} + 2e - 2.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_1^2 \frac{v^3 + 3v^6}{v^4}\, dv &= \int_1^2 \frac{1}{v} + 3v^2\, dv \\
&= \ln|v| + v^3\Big|_1^2 \\
&= \left(\ln|2| + (2)^3\right) - \left(\ln|1| + (1)^3\right) \\
\int_1^2 \frac{v^3 + 3v^6}{v^4}\, dv &= \ln(2) + 7.
\end{aligned}$$

(c) Indefinite integral – don't forget the "+C"!

$$\int (\cos(2x) - \sin(x))\, dx = \frac{1}{2}\sin(2x) + \cos(x) + C$$

(d)

$$\begin{aligned}
\int_1^8 \sqrt[3]{x}\, dx &= \int_1^8 x^{1/3}\, dx \\
&= \left.\frac{x^{4/3}}{(4/3)}\right|_1^8 \\
&= \left.\frac{3x^{4/3}}{4}\right|_1^8 \\
&= \frac{3(8)^{4/3}}{4} - \frac{3(1)^{4/3}}{4} \\
&= 12 - \frac{3}{4} \\
\int_1^8 \sqrt[3]{x}\, dx &= \frac{45}{5}.
\end{aligned}$$

When evaluating powers like $8^{4/3}$, remember that $8^{4/3} = (8^{1/3})^4 = (8^4)^{1/3}$. So the way to calculate $8^{4/3}$ efficiently is

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16.$$

(e)

$$
\begin{aligned}
\int_1^{18} \sqrt{\frac{3}{z}}\,dz &= \sqrt{3}\int_1^{18} z^{-1/2}\,dz \\
&= \sqrt{3}\frac{z^{-1/2}}{(1/2)} \\
&= 2\sqrt{3}\sqrt{z}\Big|_1^{18} \\
&= 2\sqrt{3}\left(\sqrt{18}-\sqrt{1}\right) \\
\int_1^{18} \sqrt{\frac{3}{z}}\,dz &= 2\sqrt{3}\left(3\sqrt{2}-1\right).
\end{aligned}
$$

(f) Indefinite integral – don't forget the "+C"!

$$
\begin{aligned}
\int (1-t)\left(2+t^2\right)\,dt &= \int \left(2+t^2-2t-t^3\right)\,dt \\
&= \int \left(-t^3+t^2-2t+2\right)\,dt \\
\int (1-t)\left(2+t^2\right)\,dt &= -\frac{t^4}{4}+\frac{t^3}{3}-t^2+2t+C.
\end{aligned}
$$