Practice midterm 2 SOLUTIONS

1. Find the absolute minima and maxima, if any, of the following functions: (a) $(x+2)/(x^2+5)$ on $[-4,4]$. (b) $1/(\sin(x)+\cos(x))$ on $(-\pi/4, 3\pi/4)$.

Solution:

(a) Let $f(x) = (x+2)/(x^2+5)$. First we find the critical points, i.e. the values of x where $f'(x) = 0$. We need the derivative $f'(x)$,

$$\begin{aligned} f'(x) &= \frac{(x^2+5)\frac{d}{dx}(x+2) - (x+2)\frac{d}{dx}(x^2+5)}{(x^2+5)^2} \\ &= \frac{(x^2+5) - (x+2)(2x)}{(x^2+5)^2} \\ &= -\frac{x^2+4x-5}{(x^2+5)^2}. \end{aligned}$$

Now we solve $f'(x) = 0:$

$$\begin{aligned} -\frac{x^2+4x-5}{(x^2+5)^2} &= 0 \\ x^2+4x-5 &= 0 \\ x &= -5 \text{ or } 1. \end{aligned}$$

But $x = -5$ is outside the interval $[-4,4]$.

Therefore the candidate $x-$values that maximize/minimize $f(x)$ are $x = -4, 1, 4$. (critical points AND end points).

Now we check the function value at those points: $f(-4) = -2/21$, $f(1) = 1/2$, $f(4) = 2/7$. Comparing these, the biggest is 1/2, and the smallest is $-2/21$.

Therefore, the absolute max of $(x+2)/(x^2+5)$ on $[-4,4]$ is 1/2, achieved at $x = 1$, and the absolute min is $-2/21$, achieved at $x = -4$.

(b) Let $g(x) = 1/(\sin(x)+\cos(x))$. First we find the critical points, i.e. the values of x where $g'(x) = 0$. We need the derivative $g'(x)$,

$$\begin{aligned} g'(x) &= \frac{(\sin(x)+\cos(x))\frac{d}{dx}(1) - (1)\frac{d}{dx}(\sin(x)+\cos(x))}{(\sin(x)+\cos(x))^2} \\ &= \frac{\sin(x)-\cos(x)}{(\sin(x)+\cos(x))^2} \end{aligned}$$

Now we solve $g'(x) = 0:$

$$\begin{aligned} \frac{\sin(x)-\cos(x)}{(\sin(x)+\cos(x))^2} &= 0 \\ \sin(x) &= \cos(x) \\ x &= \pi/4 \pm n\pi. \end{aligned}$$

Note that we multiplied by $\sin(x)+\cos(x)$ to get these points, which is only OK because $\sin(x)+\cos(x)=0$ only OUTSIDE the interval (i.e. at $\pi/4$, $5\pi/4$, etc).

The only critical point in the interval $(-\pi/4, 3\pi/4)$ is $x=\pi/4$.

Therefore the candidate $x-$values that maximize/minimize $f(x)$ *is* $x=\pi/4$. Why no endpoints? It's an OPEN interval. We only include endpoints for a CLOSED interval.

Now we check the function value at that point: $f(\pi/4)=1/\sqrt{2}$ - it's an absolute max or min, but how to check? Easiest way: use another point in the interval. $f(0)=1<1/\sqrt{2}=f(\pi/4)$ so $f(\pi/4)$ is an absolute max. There is no absolute min.

Therefore, the absolute maximum of $1/(\sin(x)+\cos(x))$ on $(-\pi/4, 3\pi/4)$ is $1/\sqrt{2}$, achieved at $x=\pi/4$. There is no absolute minimum.

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2. (a) Find the linearization of the function $f(x)=\sqrt{x+16}$ at $x=0$. (b) Use the linearization above to estimate the value of $\sqrt{16.8}$.

Solution:

(a) Linearization is $L(x)=f(0)+f'(0)(x-0)$. Since $f(x)=\sqrt{x+16}$, and therefore $f'(x)=1/2\sqrt{x+16}$, $f(0)=\sqrt{16}=4$ and $f'(0)=1/2\sqrt{16}=1/8$. Putting these together, $L(x)=4+x/8$.

Therefore the linearization of $f(x)=\sqrt{x+16}$ at $x=0$ is $L(x)=4+x/8$.

(b) $\sqrt{16.8}=\sqrt{0.8+16}=f(0.8)\approx L(0.8)=4+0.8/8=4+0.1=4.1$.

Thus we used the linearization in (a) to estimate $\sqrt{16.8}\approx 4.1$.

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3. Consider the function

$$f(x)=\frac{x^3}{x^2-1}$$

(a) Find all x- and y- intercepts.
(b) Find any and all asymptote. Justify your answers using limits.
(c) Find the intervals where the function is increasing or decreasing.
(d) Find and classify all local extrema.
(e) Find the intervals where the function is concave up or concave down.
(f) Find all points of inflection (if any).
(g) Sketch a graph of the function, labeling all points.

Solution:

(a) Intercepts: The origin (0,0) is the only $x-$ or $y-$ intercept.

(b) Asymptotes: $x=-1$ and $x=1$ are the vertical asymptotes. No horizontal asymptotes. Since numerator is one degree higher than the denominator, there's an oblique asymptote: $f(x)$ can be re-written as $f(x)=x+x/(x^2-1)$ and since $\lim_{x\to\infty} x/(x^2-1)=0$, the oblique asymptote is $y=x$.

(c) Intervals of increase & decrease.

The first derivative is $f'(x)=x^2(x^2-3)/(x^2-1)^2$. Critical points are at $x=\pm 1$ (where $f'(x)$ does not exist; vertical asymptote) and at $x=0, \pm\sqrt{3}$ (where $f'(x)=0$).

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \sqrt{3}$	$x > \sqrt{3}$
Test point x^*	-2	-1.5	-0.5	0.5	1.5	2
$f'(x^*)$	4/9>0	-27/25<0	-11/9<0	-11/9<0	-27/25<0	4/9>0
Behavior of $f(x)$	increasing	decreasing	decreasing	decreasing	decreasing	increasing

Therefore $f(x)$ is increasing for $x < -\sqrt{3}$ and $x > \sqrt{3}$. It's decreasing for $-\sqrt{3} < x < \sqrt{3}$.

(d) Based on the first derivative test, encapsulated in (c), $f(-\sqrt{3}) = -3\sqrt{3}/2$ is a relative minimum. (at (-1.7,-2.6)).

$f(\sqrt{3}) = 3\sqrt{3}/2$ is a relative maximum. (approximately at (1.7,2.6)).

(iii) Derivatives: $f'(x) = x^2(x^2-3)/(x^2-1)^2$, $f''(x) = 2x(x^2+3)/(x^2-1)^3$.

(iv) Critical points of $f(x)$: Critical points are at $x = \pm 1$ (where $f'(x)$ does not exist; vertical asymptote) and at $x = 0, \pm\sqrt{3}$ (where $f'(x) = 0$).

(c) Intervals of increase & decrease.

The first derivative is $f'(x) = x^2(x^2-3)/(x^2-1)^2$. Critical points are at $x = \pm 1$ (where $f'(x)$ does not exist; vertical asymptote) and at $x = 0, \pm\sqrt{3}$ (where $f'(x) = 0$).

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \sqrt{3}$	$x > \sqrt{3}$
Test point x^*	-2	-1.5	-0.5	0.5	1.5	2
$f'(x^*)$	4/9>0	-27/25<0	-11/9<0	-11/9<0	-27/25<0	4/9>0
Behavior of $f(x)$	increasing	decreasing	decreasing	decreasing	decreasing	increasing

Therefore $f(x)$ is increasing for $x < -\sqrt{3}$ and $x > \sqrt{3}$. It's decreasing for $-\sqrt{3} < x < \sqrt{3}$.

(d) $f(-\sqrt{3}) = -3\sqrt{3}/2$ is a relative minimum. (at (-1.7,-2.6)).

$f(\sqrt{3}) = 3\sqrt{3}/2$ is a relative maximum. (approximately at (1.7,2.6)).

(e) The second derivative is $f''(x) = 2x(x^2+3)/(x^2-1)^3$. $f''(x) = 0$ at $x = 0$. HOWEVER since $x = 0$ is also a critical point, we can't say for sure that it's an inflection point, have to check concavity. $f''(x)$ does not exist at $x = \pm 1$ (vertical asymptotes) - these should be checked, too.

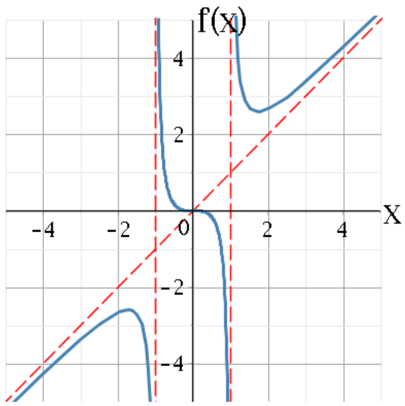
	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
Test point x^*	-2	-1/2	1/2	2
$f''(x^*)$	-28/27<0	208/27>0	-208/27	28/27>0
Behavior of $f(x)$	concave down	concave up	concave down	concave up

$f(x)$ is concave down for $x < -1$ and $0 < x < 1$.

$f(x)$ is concave up for $-1 < x < 0$ and $x > 1$.

(f) $x = 0$ is an inflection point. Concavity also changes across asymptotes, at $x = \pm 1$.

(g) Sketch... Connect the dots.



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4. Use implicit differentiation to find the equation of the tangent line to the following curve at the point $(1,1)$:

$$x\sin(xy-y^2)=x^2-1.$$

Solution: First we calculate the derivative

$$\begin{aligned}\frac{d}{dx}\left(x\sin(xy-y^2)\right) &= \frac{d}{dx}\left(x^2-1\right)\\ \sin(xy-y^2)+x\cos(xy-y^2)\left(y+x\frac{dy}{dx}-2y\frac{dy}{dx}\right) &= 2x\\ \frac{dy}{dx} &= \frac{xy\cos(xy-y^2)+\sin(xy-y^2)-2x}{x(2y-x)\cos(xy-y^2)}.\end{aligned}$$

For the tangent line to the curve at $(1,1)$ we require the slope, which is the derivative evaluated at (1,1):

$$\begin{aligned}m &= \left.\frac{dy}{dx}\right|_{x=y=1}\\ &= \frac{\cos(0)+\sin(0)-2}{(2-1)\cos(0)}\\ &= -1.\end{aligned}$$

Now we use the point-slope formula. The equation of the line is

$$y-y_1=m(x-x_1)$$

where $(x_1,y_1)=(1,1)$ and $m=-1$. Therefore

$$\begin{aligned}y-1 &= -(1)(x-1)\\ y-1 &= -x+1\\ y &= -x+2.\end{aligned}$$

Therefore the equation for the tangent line to the curve is

$$y = -x + 2.$$

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5. At 12:00 noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?

Solution: Related rates! And aren't sketches helpful... (mine's on the back of an envelope).

Let B's original position be the origin, with B traveling along the $x-$axis and A traveling along the $y-$axis.

Let x be the position of ship A relative to B's original position (origin). After 4 hours, ship A has sailed 35 km/h $\times$ 4 hours = 140 km, east. It is now 10 km west of B's original position. $x = -10$ km, and dx/dt=35 km/h (all along the $x-$axis).

Let y be the distance of ship B from its original position (origin). After 4 hours, ship B has sailed 25 km/h $\times$ 4 hours = 100 km, north. It is now 100 km north of its original position. $y = 100$ km, and dy/dt=25 km/h (all along the $y-$axis).

We can use the Pythagorean identity to describe the distance D between the two ships:

$$D^2 = x^2 + y^2.$$

After 4 hours, the distance between the ships is $D = \sqrt{x^2 + y^2} = \sqrt{(-10\text{ km})^2 + (100\text{ km})^2} = 10\sqrt{101}$ km.

We want the rate of change of the distance between the two, dD/dt. Take the time derivative,

$$\begin{aligned} \frac{d}{dt}\left(D^2\right) &= \frac{d}{dt}\left(x^2 + y^2\right) \\ 2D\frac{dD}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{dD}{dt} &= \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right). \end{aligned}$$

After 4 hours,

$$\begin{aligned} \frac{dD}{dt} &= \frac{1}{10\sqrt{101}\text{ km}}\left((-10\text{ km})(35\text{ km/h}) + (100\text{ km})(25\text{ km/h})\right) \\ &= \frac{215}{\sqrt{101}}\text{ km/h}. \end{aligned}$$

Answer: At 4:00pm, the ships are moving apart at a rate of $\frac{215}{\sqrt{101}}$ km/h.

Alternative solution:

Let B's original position be the origin. Then as time t advances, in hours, the position of ship A is given by $x(t) = 35t - 150$, along the $x-$axis. And the position of ship B is given by $y(t) = 25t$, along the $y-$axis.

The distance between the ships is given by $\ell(t) = \sqrt{x(t)^2 + y(t)^2} = \sqrt{(35t-150)^2 + 25^2} = 5\sqrt{74t^2 - 420t + 900}$ (in kilometers). We want the rate of change in this distance at $t = 4$ hours. That is, we want $\frac{d\ell}{dt}\big|_{t=4}$.

$$\begin{aligned}\frac{d\ell}{dt} &= \frac{d}{dt}\left(5\sqrt{74t^2 - 420t + 900}\right) \\ &= \frac{10(37t-105)}{\sqrt{74t^2 - 420t + 900}}.\end{aligned}$$

Then evaluating at $t = 4$ hours,

$$\left.\frac{d\ell}{dt}\right|_{t=4} = \frac{215}{\sqrt{101}} \text{ km/h.}$$

Answer: At 4:00pm, the ships are moving apart at a rate of $\frac{215}{\sqrt{101}}$ km/h.

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6. An open box is to be made by cutting a square from each corner of a 12-in. by 12-in. piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?

Solution:

Consult the sketch. Note that since $12 - 2x > 0$, $x < 6$ (at $x = 6$, after cutting away the squares, there's nothing left!). The volume of the box is given by

$$\begin{aligned}V &= (12-2x)^2 x \\ &= x(144 - 48x + 4x^2) \\ &= 144x - 48x^2 + 4x^3.\end{aligned}$$

We want to maximize this volume.

First find the critical points of the volume function, which are given by solutions of $V'(x) = 0$. The first derivative $V'(x)$ is

$$\begin{aligned}V'(x) &= 144 - 96x + 12x^2 \\ &= 12(12 - 8x + x^2) \\ &= 3(x-2)(x-6).\end{aligned}$$

Solving $V'(x) = 0$ we find the critical points are $x = 2$ and $x = 6$. But $x = 6$ is outside the interval so the only critical point is $x = 2$.

Next we determine if the critical point $x = 2$ maximizes or minimizes the volume of the box. We use the second derivative test. The second derivative is

$$V''(x) = -96 + 24x.$$

Evaluating at the critical points, $V''(2) = -96 + 24(2) = -48 < 0$. $V(2)$ is therefore a maximum.

Answer: To maximize the volume of the box, cut a 2 in $\times$ 2 in square out of each of the corners.

Note: If you included the $x = 6$ in the interval of consideration, the second derivative test would reveal it to be a minimum (volume $V = 0!$), so you'd still get the right answer.

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7. A ladder 15 feet long tests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fact is the top of the ladder sliding down the wall when the bottom of the ladder is 9 ft from the wall?

Solution: Related rates problem!

Let $h = h(t)$ be the height of the top of the ladder against the wall and $x = x(t)$ be the distance of the base of the ladder from the wall. We are given $dx/dt = 1$ ft/sec and are asked for dh/dt when $x = 9$ ft.

To relate x and h use the Pythagorean identity,

$$h^2 + x^2 = (15 \text{ ft})^2.$$

Then take a derivative,

$$\begin{aligned} \frac{d}{dt}\left(h^2 + x^2\right) &= \frac{d}{dt}\left((15 \text{ ft})^2\right) \\ 2h\frac{dh}{dt} + 2x\frac{dx}{dt} &= 0 \\ \frac{dh}{dt} &= -\frac{x}{h}\frac{dx}{dt}. \end{aligned}$$

To find dh/dt we need dx/dt, x, and h. We know x and dx/dt, need to find h. When $x = 9$ ft,

$$\begin{aligned} h^2 + x^2 &= (15 \text{ ft})^2 \\ h^2 + (9 \text{ ft})^2 &= (15 \text{ ft})^2 \\ h^2 &= 144 \text{ ft}^2 \\ h &= 12 \text{ ft}. \end{aligned}$$

Now we can calculate dh/dt:

$$\begin{aligned} \frac{dh}{dt} &= -\frac{x}{h}\frac{dx}{dt} \\ &= -\frac{(9 \text{ ft})}{(12 \text{ ft})}(1 \text{ ft/sec}) \\ &= -\frac{3}{4} \text{ ft/sec}. \end{aligned}$$

Answer: When the base is 9 ft from the wall, the height is dropping at the rate of $-3/4$ ft/sec.

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8. Jack and Jill have an on-again off-again love affair. The sum of their love for one another is given by the function $y(t) = \sin(2t) + \cos(2t)$. (a) Find the times when their total love is at a maximum. (b) Find the times when they dislike each other the most.

Solution: Chatter about Jack and Jill aside, we're asked for the maxima and minima of

$$y(t) = \sin(2t) + \cos(2t).$$

Find the critical points. These are solutions of $y'(t) = 0$:

$$\begin{aligned} y'(t) &= 0 \\ \frac{d}{dt}(\sin(2t) + \cos(2t)) &= 0 \\ 2\cos(2t) - 2\sin(2t) &= 0 \\ \sin(2t) &= \cos(2t) \\ \tan(2t) &= 1 \\ 2t &= \frac{\pi}{4} + n\pi \\ t &= \frac{\pi}{8} + \frac{n\pi}{2}, \end{aligned}$$

where n is an integer.

Next we verify if critical points represent maxima or minima, using the second derivative test. The second derivative is

$$\begin{aligned} y''(t) &= \frac{d}{dt}(2\cos(2t) - 2\sin(2t)) \\ &= -4\sin(2t) - 4\cos(2t) \end{aligned}$$

Testing the critical points,

$$\begin{aligned} y''\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) &= -4\sin\left(2\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)\right) - 4\cos\left(2\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)\right) \\ &= -4\sin\left(\frac{\pi}{4} + n\pi\right) - 4\cos\left(\frac{\pi}{4} + n\pi\right). \end{aligned}$$

For n even, $\pi/4 + n\pi$ is in the **first** quadrant so

$$y''\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = -4\sin\left(\frac{\pi}{4} + n\pi\right) - 4\cos\left(\frac{\pi}{4} + n\pi\right) = -4\sin\left(\frac{\pi}{4}\right) - 4\cos\left(\frac{\pi}{4}\right) = -4\sqrt{2} < 0.$$

Therefore the critical points $t = \pi/8 + n\pi/2$, for n even, are relative maxima. That is, $t = \pi/8$, $9\pi/8$, $17\pi/8$, ... are relative maxima. And since $y\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = 1$ regardless of n so long as n is even, $y\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = 1$ is the absolute maximum, achieved at $t = \pi/8$, $9\pi/8$, $17\pi/8$...

For n odd, $\pi/4 + n\pi$ is in the **third** quadrant so

$$y''\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = -4\sin\left(\frac{\pi}{4} + n\pi\right) - 4\cos\left(\frac{\pi}{4} + n\pi\right) = -4\sin\left(\frac{5\pi}{4}\right) - 4\cos\left(\frac{5\pi}{4}\right) = 4\sqrt{2} > 0.$$

Therefore the critical points $t = \pi/8 + n\pi/2$, for n odd, are relative minima. That is, $t = 5\pi/8$, $13\pi/8$, $21\pi/8$, ... are relative minima. And since $y\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = -1$ regardless of n so long as n is odd, $y\left(\frac{\pi}{8} + \frac{n\pi}{2}\right) = -1$ is the absolute minimum, achieved at $t = 5\pi/8$, $13\pi/8$, $21\pi/8$, ...

Answer: (a) Jack and Jill's total love is at a maximum at times $t = \pi/8 + n\pi/2$, for n **even**. (b) Jack and Jill's total love is at a minimum at times $t = \pi/8 + n\pi/2$, for n **odd**.
