Chapter 6

What the derivative tells us about a function

The derivative of a function contains a lot of important information about the behaviour of a function. In this chapter we will focus on how properties of the first and second derivative can be used to help up refine curve-sketching techniques.

6.1 Overall shape of the graph of a function

Section 6.1 Learning goals

1. Understand that the sign of the first derivative corresponds to an increasing or decreasing property of a function.
2. Understand that the sign of the second derivative correspond to the concavity (curvature) of a function.

6.1.1 Increasing and decreasing functions

Consider a function given by $y = f(x)$. We first make the following observations:

1. If $f'(x) > 0$ then $f(x)$ is *increasing*.
2. If $f'(x) < 0$ then $f(x)$ is *decreasing*.

Naturally, we read graphs from left to right, i.e. in the direction of the positive x axis, so when we say "increasing" we mean that as we move from left to right, the value of the function gets larger.

We can use the same ideas to relate the second derivative to the first derivative.

1. If $f''(x) > 0$ then $f'(x)$ is **increasing**. This means that the slope of the original function is getting steeper (from left to right). The function curves upwards: we say that it is *concave up*. See Figure 6.1(a).

2. If $f''(x) < 0$ then $f'(x)$ is **decreasing**. This means that the slope of the original function is getting shallower (from left to right). The function curves downwards: we say that it is *concave down*. See Figure 6.1(b).

6.1.2 Concavity and points of inflection

The second derivative of a function provides information about the **curvature** of the graph of the function, also called the **concavity** of the function. In Figure 6.1(a), $f(x)$ is **concave up**, and its second derivative (not shown) would be positive. In Figure 6.1(b), $f(x)$ is **concave down**, and second derivative would be negative.

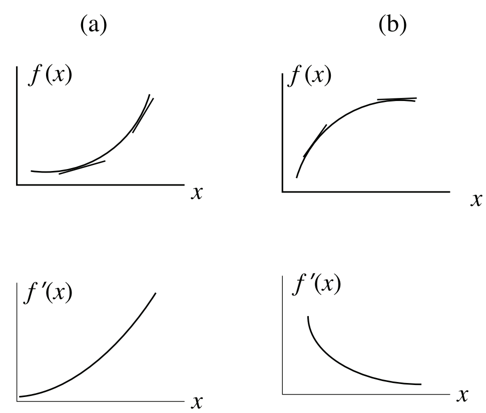

Figure 6.1. *In (a) the function is concave up, and its derivative thus increases (in the positive direction). In (b), for a concave down function, we see that the derivative decreases.*

Definition 6.1. *A* **point of inflection** *of a function $f(x)$ is a point x at which the concavity of the function changes. (See, for example, Fig. 6.2.)*

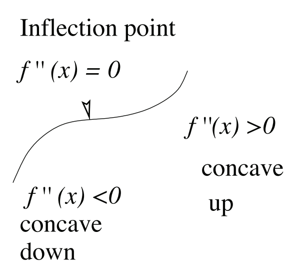

Figure 6.2. *An inflection point is a place where the concavity of a function changes.*

We can deduce from the definition and previous remarks that at a point of inflection the *second derivative changes sign*. This is illustrated in Figure 6.2. **Note carefully:** *It is not enough to show that $f''(x) = 0$ to conclude that x is an inflection point.* We summarize the one-way nature of this relationship in the box. Then, in Example 6.2 we show why this is true.

Inflection points

1. If the function $y = f(x)$ has a point of inflection at x_0 then $f''(x_0) = 0$.
2. If the function $y = f(x)$ satisfies $f''(x_0) = 0$, we **cannot conclude** that it has a point of inflection at x_0. We must actually check that $f''(x)$ changes sign at x_0.

Example 6.2 Consider the the functions (a) $f_1(x) = x^3$ and (b) $f_2(x) = x^4$. Show that for both functions, the second derivative is zero at the origin ($f''(0) = 0$) but that only one of these functions actually has an inflection point at $x = 0$. ■

Solution: The functions are

$$\text{(a) } y = f_1(x) = x^3, \quad \text{(b) } y = f_2(x) = x^4.$$

The first derivatives are

$$\text{(a) } y = f_1'(x) = 3x^2, \quad \text{(b) } y = f_2'(x) = 4x^3.$$

and the second derivatives are:

$$\text{(a) } y = f_1''(x) = 6x, \quad \text{(b) } y = f_2''(x) = 12x^2.$$

Thus, at $x = 0$ we have $f_1''(0) = 0, f_2''(0) = 0$. However, $x = 0$ is **NOT** an inflection

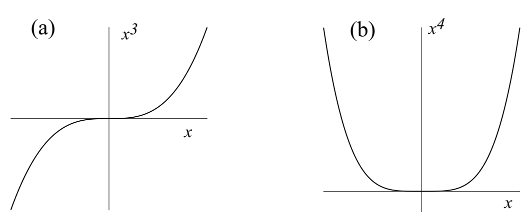

Figure 6.3. *The functions (a) $f_1(x) = x^3$ and (b) $y = f_2(x) = x^4$ both satisfy $f''(0) = 0$. However, only x^3 has an inflection point at $x = 0$, whereas x^4 has a local minimum at that point. This results from the fact that $f_2''(x)$ does not change sign at $x = 0$.*

point of x^4. In fact, it is a local minimum, as is evident from Figure 6.3.

6.1.3 Determining whether $f''(x)$ changes sign

In the previous section, we defined an inflection point as a point on the graph of a function at which the second derivative changes sign. But how do we detect if this sign change occurs at a given point? Here we address this question and provide a few helpful tools.

We first state the following important result

Sign change in a product of factors:
If an expression is a product of factors, e.g. $g(x) = (x - a_1)^{n_1}(x - a_2)^{n_2} \dots (x - a_m)^{n_m}$, then

1. The expression can be zero only at the points $x = a_1, a_2, \dots, a_m$.
2. The expression changes sign only at points $x = a_i$ for which n_i is an odd integer power.

Example 6.3 Determine where the expression $g(x) = x^2(x + 2)(x - 3)^4$ changes sign. ■

Solution: The zeros of $g(x)$ are $x = 0, -2, 3$. However, $g(x)$ only changes sign at $x = -2$. Close to this point, $g(x) \approx g(-2) \approx (-2)^2(x + 2)(-5)^4 = 2500(x + 2)$. Clearly for $x < -2$ this is negative and for $x > -2$, this is positive. Hence there is a sign change at $x = -2$. At $x = 0$ and at $x = 3$ there is no sign change as the terms x^2 and $(x - 3)^4$ are always positive.

Example 6.4 Find all inflection points of the function $f(x) = (2/5)x^6 - x^4 + x$. ■

Solution: We compute the derivatives of the function, and find these to be

$$f'(x) = (12/5)x^5 - 4x^3 + 1, \quad f''(x) = 12x^4 - 14x^2 = 12x^2(x^2 - 1) = 12x^2(x + 1)(x - 1).$$

Here we have completely factorized the second derivative so that it would be easy to identify factors with even and odd powers, to find locations where the second derivative changes sign. We see that there is NO sign change at $x = 0$, whereas at both $x = -1, 1$ there is a sign-changing factor. Thus the infection points are at $x = -1, 1$.

6.2 Special points on the graph of a function

In this section we use tools of algebra and calculus to identify special points on the graph of a function. We first consider the **zeros** of a function, and then its critical points.

Section 6.2 Learning goals

1. Understand the definition of a zero of a function and be able to identify zeros for simple functions (factorizable polynomials).
2. Understand that a function $f(x)$ can have various types of critical points (maxima, minima, and other types) at which $f'(x) = 0$.
3. Be able to find critical points for a given function.
4. Using first or second derivative tests, be able to classify a given critical point as a maximum, minimum (or neither).

6.2.1 Zeros of a function

Definition 6.5 (Zero). *Given a function $y = f(x)$, we say that x_0 is a* **zero** *of f if $f(x_0) = 0$. In this case we also say that "x_0 is a* **root** *of the equation $f(x) = 0$".*

Example 6.6 (Factoring) Find the zeros of the function $y = f(x) = x^2 - 5x + 6$. ■

Solution: This function is a polynomial that factors into $f(x) = (x - 3)(x - 2)$. Thus we look for values of x satisfying $0 = (x - 3)(x - 2)$. We use the fact that when a product of factors is zero, at least one of the factors must be zero. This means that either $(x - 3) = 0$ or $(x - 2) = 0$, so $x = 2, 3$ are the two zeros of the function.

Example 6.7 Find zeros of the function $y = f(x) = x^3 - 3x^2 + x$. ■

Solution: We can factor this function into $f(x) = x(x^2 - 3x + 1)$. From this we see that $x = 0$ is one of the desired zeros of f. To find the others, we use the quadratic formula on the second factor, obtaining

$$x_{1,2} = \frac{1}{2}(3 \pm \sqrt{3^2 - 4}) = \frac{1}{2}(3 \pm \sqrt{5}).$$

Thus, there is a total of three zeros in this case.

Example 6.8 Find zeros of the function $y = f(x) = x^3 - x - 3$. ■

Solution: This polynomial does not factor, nor is it easy to apply a cubic formula (analogous to the quadratic formula) for such cases. Rather than use such a formula, we will give up the elementary algebraic techniques, and use an approximation method, to be discussed later in Example 5.10.

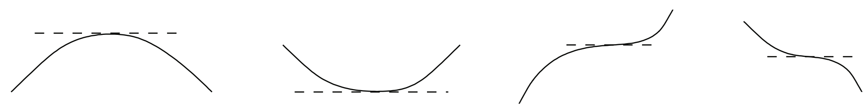

Figure 6.4. *A critical point (place where $f'(x) = 0$) can be a local maximum, local minimum, or neither.*

6.2.2 Critical points

Definition 6.9. *A* **critical point** *of the function $f(x)$ is any point x at which the first derivative is zero, i.e. $f'(x) = 0$.*

Clearly, this will occur whenever the slope of the tangent line to the graph of the function is zero, i.e. the tangent line is horizontal. Figure 6.4 shows several possible shapes of the graph of function close to a critical point.

We will call the first of these (on the left) a **local maximum**, the second a **local minimum**, and the last two cases (which are bends in the curve) inflection points.

In many scientific applications, critical points play a very important role. (We will see examples of this sort shortly.) We would like some criteria for determining whether a critical point is a local maximum, minimum, or neither. We will develop such diagnoses in the next section.

Example 6.10 Consider the function $y = f(x) = x^3 + 3x^2 + ax + 1$. For what range of values of a are there no critical points? ■

Solution: We compute the first derivative $f'(x) = 3x^2 + 6x + a$. A critical point would occur whenever $0 = f'(x)$, which implies $0 = 3x^2 + 6x + a$. This is a quadratic equation whose solutions are

$$x_{1,2} = \frac{-6 \pm \sqrt{36 - 4a \cdot 3}}{6}.$$

This leads to two real solutions *unless* $36 - 12a < 0$. In that case, there are no real solutions. Thus there will be no critical points when $36 - 12a < 0$, which corresponds to $a > 3$.

6.2.3 What happens close to a critical point

From Figure 6.5 we see the behaviour of the first and second derivatives of a function close to critical points. We already know that at the point in question, $f'(x) = 0$, so clearly the graph of $f'(x)$ crosses the x axis at each critical point. However, note that next to a local maximum, (and reading from left to right, as is the convention in any graph) the slope of $f(x)$ is first positive (to the left), then becomes zero (at the critical point) and then becomes negative (to the right of the point). This means that the derivative is *decreasing* from left to right, as indicated in Figure 6.5.

Since the changes in the first derivative are measured by *its* derivative, i.e. by $f''(x)$, we can say, equivalently that the second derivative is negative at a local maximum.

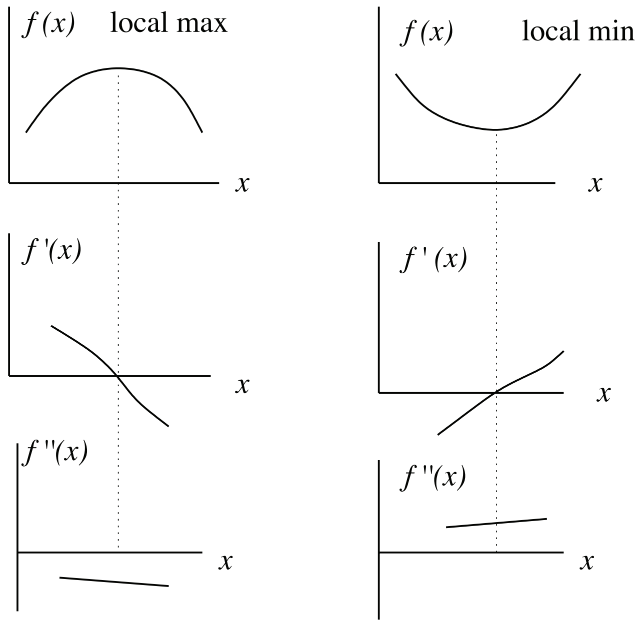

Figure 6.5. *Close to a local maximum, $f(x)$ is concave down, $f'(x)$ is decreasing, so that $f''(x)$ is negative. Close to a local minimum, $f(x)$ is concave up, $f'(x)$ is increasing, so that $f''(x)$ is positive.*

The converse is true near any local minimum. This is shown on the right column of Figure 6.5. We conclude from this discussion that the following diagnosis would distinguish a local maximum from a local minimum:

Summary: first derivative

$f'(x) < 0$	$f'(x_0) = 0$	$f'(x) > 0$
decreasing function	critical point at x_0	increasing function

Summary: second derivative

$f''(x) < 0$	$f''(x_0) = 0$	$f''(x) > 0$
curve concave down	check for inflection point at x_0 if f'' changes sign	curve concave up

Summary: type of critical point

- **First derivative test**: This test depends on the way that the *sign* of the first derivative changes close to the critical point. Near a **local maximum**, the first derivative has a transition from positive to zero to negative values reading across the graph from left to right, as shown in the middle left panel of Fig. 6.5 and the table below:

$x < x_0$ $f'(x) > 0$	$x = x_0$ $f'(x_0) = 0$	$x > x_0$ $f'(x) < 0$

Near a **local minimum**, the first derivative goes from negative to zero to positive values as shown in the middle right panel of Fig. 6.5 and the table below:

$x < x_0$ $f'(x) < 0$	$x = x_0$ $f'(x_0) = 0$	$x > x_0$ $f'(x) > 0$

- **Second derivative test**: At a local maximum, the second derivative is negative. At a local minimum, the second derivative is positive.

Here we assume that x_0 is a critical point, i.e. a point at which $f'(x_0) = 0$. Then the following table summarizes what happens at that point

$f''(x_0) < 0$	$f''(x_0) = 0$	$f''(x_0) > 0$
local maximum	inconclusive	local minimum

Inflection points:

We look for points at which $f''(x_0) = 0$ and check that f'' changes sign at x_0. When both these conditions are satisfied, we conclude that x_0 is an inflection point.

6.3 Sketching the graph of a function

Recall that in Section 1.4, we used elementary reasoning about power functions to sketch the graph of simple polynomials. Now that we have at our disposal more advanced calculus techniques, we will be able to hone such methods to produce more detailed and more accurate sketches of the graph of a function. We devote this section to illustrating these new methods and their application.

Section 6.3 Learning goals

1. Given a function (polynomial, rational function, etc) be able to find its zeros, critical points, inflection points, and determine where it is increasing or decreasing, concave up or down.
2. Using a combination of the above techniques, together with methods of Section 1.4, assemble a reasonably accurate sketch of the graph of the function.
3. Using these techniques, be able to identify all local as well as global **extrema** (minima and maxima) of a function $f(x)$ on an interval $a \leq x \leq b$.

Example 6.11 Sketch the graph of the function $B(x) = C(x^2 - x^3)$. ∎

Solution: To prepare the way, we compute the derivatives:

$$B'(x) = C(2x - 3x^2), \quad B''(x) = C(2 - 6x).$$

The following set of steps will be a useful way to proceed:

1. We can easily find the *zeros* of the function by setting $B(x) = 0$. We find that
$$C(x^2 - x^3) = 0, \quad \Rightarrow \quad x^2 = x^3$$
so $x = 0$ or $x = 1$ are the solutions.

2. By considering powers, we note that close to the origin, the power x^2 would dominate (so we expect to see something resembling a parabola opening upwards close to the origin), whereas, far away, where the term $-x^3$ dominates, we expect an (upside down) cubic curve, as shown in a preliminary sketch in Figure 6.6.

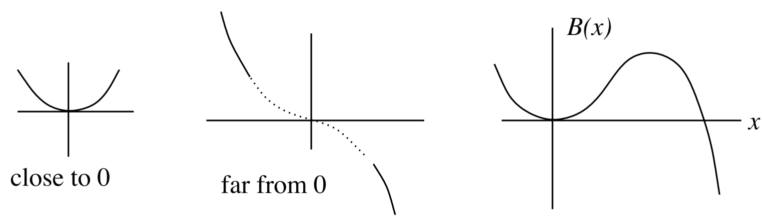

Figure 6.6. *Figure for the function $B(x) = C(x^2 - x^3)$ in Example 6.11 showing which power dominates.*

3. To find the critical points, we set $B'(x) = 0$, obtaining
$$B'(x) = C(2x - 3x^2) = 0, \quad \Rightarrow \quad 2x - 3x^2 = 0, \quad \Rightarrow \quad 2x = 3x^2$$
so either $x = 0$ or $x = 2/3$. From the sketch in Figure 6.6 it is clear that the first is a local minimum, and the second a local maximum. (But we will also get a confirmation of this fact from the second derivative.)

4. From the second derivative we find that $B''(0) = 2 > 0$ so that $x = 0$ is indeed a local minimum. Further, $B''(2/3) = 2 - 6 \cdot (2/3) = -2 < 0$ so that $x = 2/3$ is a local maximum. This is the confirmation that our sketch makes sense.

5. Now identifying where $B''(x) = 0$, we find that
$$B''(x) = C(2 - 6x) = 0, \quad \text{when} \quad 2 - 6x = 0 \quad \Rightarrow \quad x = \frac{2}{6} = \frac{1}{3}$$
we also note that the second derivative changes sign here: i.e. for $x < 1/3$, $B''(x) > 0$ and for $x > 1/3$, $B''(x) < 0$. Thus there is an inflection point at $x = 1/3$. The final sketch would be as given in Figure 6.7.

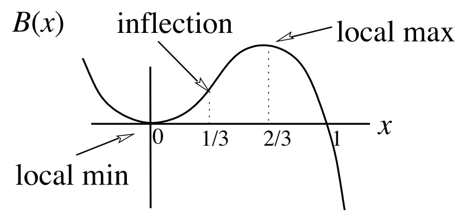

Figure 6.7. *Figure for the function $B(x) = C(x^2 - x^3)$ in Example 6.11.*

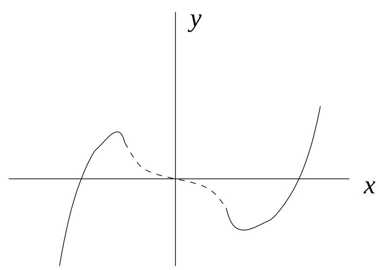

Figure 6.8. *The function $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$ of Example 6.12 behaves roughly like the negative cubic near the origin, and like $8\,x^5$ for large x.*

Example 6.12 Sketch the graph of the function $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$ ■

Solution:

1. **Consider the powers:**

   The highest power is $8\,x^5$ so that far from the origin we expect a typical positive odd function behavior.

   The lowest power is $-20\,x^3$, which means that close to zero, we would expect to see a negative cubic. This already indicates to us that the function "turns around", and so, must have some local maxima and minima. We draw a rough sketch in Figure 6.8.

2. **Zeros:** Factoring the expression for y leads to

   $$y = x^3(8x^2 + 5x - 20).$$

   Using the quadratic formula, we can find the places where $y = 0$, i.e. the *zeros* of the function. They are

   $$x = 0,\ 0,\ 0,\ -\frac{5}{16} + \frac{1}{16}\sqrt{665},\ -\frac{5}{16} - \frac{1}{16}\sqrt{665}$$

   In decimal form, these are approximately $x = 0, 0, 0, 1.3, -1.92$

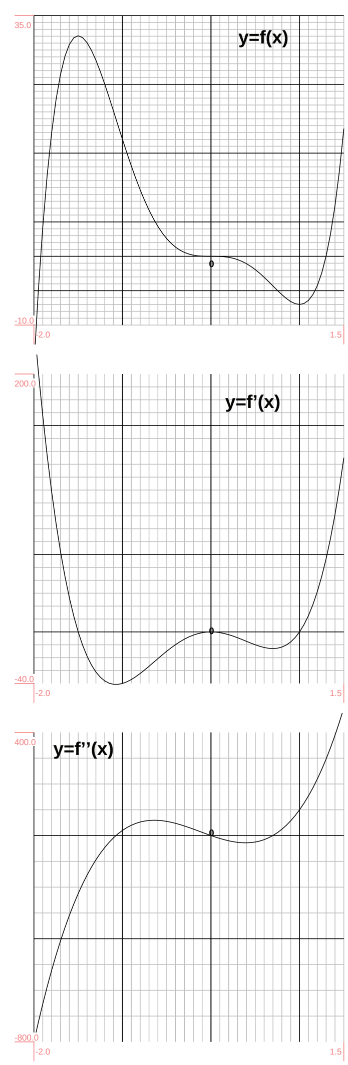

Figure 6.9. *The function* $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$*, and its first and second derivatives,* $f'(x)$ *and* $f''(x)$

3. **First derivative:** Calculating the derivative of $f(x)$ and then factoring leads to

$$\frac{dy}{dx} = f'(x) = 40\,x^4 + 20\,x^3 - 60\,x^2 = 20x^2(2x+3)(x-1)$$

so that the places where this derivative is zero are: $x = 0, 0, 1, -3/2$. We expect critical points at these places.

4. **Second derivative:** We calculate the second derivative and factor to obtain

$$\frac{d^2y}{dx^2} = f''(x) = 160\,x^3 + 60\,x^2 - 120\,x = 20x(8x^2 + 3x - 6)$$

Thus, we can find places where the second derivative is zero. This occurs at

$$x = 0,\ -\frac{3}{16} + \frac{1}{16}\sqrt{201},\ -\frac{3}{16} - \frac{1}{16}\sqrt{201}$$

The values of these roots can be approximated by: $x = 0, 0.69, -1.07$

5. **Classifying the critical points:** To identify the types of critical points, we can use the second derivative test, i.e. determine the sign of the second derivative at each of the critical points.

   At $x = 0$ we see that $f''(0) = 0$ so the test is inconclusive. At $x = 1$, we have $f''(1) = 20(8 + 3 - 6) > 0$ implying that this is a local minimum. At $x = -3/2$ we have $f''(-1.5) = -225 < 0$ so this is a local maximum. In fact we find that the value of the function at $x = -1.5$ is $y = f(-1.5) = 32.0625$, whereas at $x = 1$ $f(1) = -7$.

The table below summarizes what we have found, and what we concluded. Each of the values of x across its top row has some significance in terms of the behaviour of the function.

$x =$	-1.92	-1.5	-1.07	0	0.69	1	1.3
$f(x) =$	0	32.0		0		-7	0
$f'(x) =$		0		0		0	
$f''(x) =$		< 0	0	0	0	> 0	
	zero	max	inflection		inflection	min	zero

We can now sketch the shape of the function, and its first and second derivatives in Figure 6.9.

6.3.1 Global maxima and minima, endpoints of an interval

Global (absolute) maxima and minima:

A global (or absolute) maximum of a function $y = f(x)$ over some interval is the largest value that the function attains on that interval. Similarly a global (or absolute) minimum is the smallest value.

Comment: If the function is defined on a closed interval, we must check both the local maxima and minima as well as the endpoints of the interval to determine where the global maxima and minima occur.

Example 6.13 Consider the function $y = f(x) = \frac{2}{x} + x^2 \quad 0.1 < x < 4$. Find the largest and smallest values that this function takes over the given interval. ∎

Solution: We first compute the derivatives:

$$f'(x) = -2\frac{1}{x^2} + 2x,$$

$$f''(x) = 4\frac{1}{x^3} + 2.$$

We now determine where critical points $f'(x) = 0$ occur:

$$-2\frac{1}{x^2} + 2x = 0.$$

Simplifying, we find $-2\frac{1}{x^2} = 2x$, so $x^3 = 1$ and the critical point is at $x = 1$. Observe that the second derivative at this point is

$$f''(1) = 4\frac{1}{1^3} + 2 = 6 > 0,$$

so that $x = 1$ is a local minimum.

We now calculate the value of the function at the endpoints $x = 0.1$ and $x = 4$ as well as at the critical point $x = 1$ to determine where global and local minima and/or maxima occur:

$f(0.1) = 20.01$	$f(1) = 3$	$f(4) = 16.5$
global maximum	global minimum	

We see that the global minimum occurs at $x = 1$. There are no local maxima. The global maximum occurs at the left endpoint.