# INDETERMINATE FORM OF TYPE $\infty - \infty$
# EXAMPLES FROM SECTION 7.8

Indeterminate forms of type $\infty - \infty$ can be the hardest to evaluate. Often you'll find either putting the expression on a common denominator, rationalizing, conjugates, or factoring something out to reduce the difference to a single fraction or product.

**(7) Evaluate**

$$\lim_{x\to 1^+}\left(\frac{x}{1-x}-\frac{1}{\ln x}\right).$$

**SOLUTION:**

Since $\lim_{x\to 1^+} x/(1-x) = \infty$ and $\lim_{x\to 1^+} 1/\ln x = \infty$, we have an indeterminate form of type $\infty - \infty$.

Indeterminate differences require some ingenuity to evaluate the limit.

In this case we try putting the expression on a common denominator:

$$\lim_{x\to 1^+}\left(\frac{x}{1-x}-\frac{1}{\ln x}\right) = \lim_{x\to 1^+}\left(\frac{1-x(1+\ln x)}{(x-1)\ln x}\right),$$

after some simplification. Now,

$$\begin{aligned} \lim_{x\to 1^+} 1-x(1+\ln x) &= 0 \\ \lim_{x\to 1^+} (x-1)\ln x &= 0, \end{aligned}$$

so we have an indeterminate form of type $0/0$. We can therefore use L'Hospital's rule,

$$\begin{aligned} \lim_{x\to 1^+}\left(\frac{x}{1-x}-\frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\left(\frac{1-x(1+\ln x)}{(x-1)\ln x}\right) \\ &\overset{H}{=} \lim_{x\to 1^+}\left(\frac{-\ln x-2}{\ln x-1/x+1}\right) \\ &= -\infty. \end{aligned}$$

(noting that as $x \to 1^+$, $\ln(x) > 0$).

Therefore your final answer is

$$\lim_{x\to 1^+}\left(\frac{x}{1-x}-\frac{1}{\ln x}\right) = -\infty.$$

**Another example. Evaluate**

$$\lim_{x\to\infty}\sqrt{x^2-x}-x$$

**SOLUTION:**

Factor out an $'x'$:

$$\lim_{x\to\infty}\sqrt{x^2-x}-x = \lim_{x\to\infty} x\left(\sqrt{1-1/x}-1\right).$$

Now note that

$$\begin{aligned}\lim_{x\to\infty} x &= \infty \\ \lim_{x\to\infty}(\sqrt{1-1/x}-1) &= 0,\end{aligned}$$

so we have an indeterminate form of type $0\times\infty$. Next we re-write as

$$\lim_{x\to\infty}\sqrt{x^2-x}-x = \lim_{x\to\infty}\frac{\left(\sqrt{1-1/x}-1\right)}{1/x},$$

where

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{1-1/x}-1\right) &= 0 \\ \lim_{x\to\infty}(1/x) &= 0,\end{aligned}$$

so we have an indeterminate form of type $0/0$. We can therefore use L'Hospital's rule,

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2-x}-x &= \lim_{x\to\infty}\left(\frac{\sqrt{1-1/x}-1}{1/x}\right) \\ &\stackrel{H}{=} \lim_{x\to\infty}\left(\frac{1/\left(2x^2\,(1-1/x)^{-1/2}\right)}{-1/x^2}\right) \\ &= \lim_{x\to\infty}\left(-\frac{1}{2\sqrt{1-1/x}}\right) \\ &= -\frac{1}{2}.\end{aligned}$$

Thus we have shown that

$$\lim_{x\to\infty}\sqrt{x^2-x}-x = -\frac{1}{2}.$$

*Alternative approach:* Instead of factoring, we could have multiplied by a conjugate,

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2-x}-x &= \lim_{x\to\infty}\left(\sqrt{x^2-x}-x\right)\times\left(\frac{\sqrt{x^2-x}+x}{\sqrt{x^2-x}+x}\right) \\ &= \lim_{x\to\infty}-\frac{x}{\sqrt{x^2-x}+x},\end{aligned}$$

which since

$$\begin{aligned}\lim_{x\to\infty} x &= \infty \\ \lim_{x\to\infty}(\sqrt{x^2-x}+x) &= \infty,\end{aligned}$$

is an indeterminate form of the type $\infty/\infty$, and we can use L'Hospital's rule,

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2-x}-x &= \lim_{x\to\infty}-\frac{x}{\sqrt{x^2-x}+x}\\ &\overset{H}{=} \lim_{x\to\infty}-\frac{1}{(2x-1)/\left(2\sqrt{x^2-x}\right)+1}.\end{aligned}$$

Then since

$$\lim_{x\to\infty}\left(\frac{2x-1}{2\sqrt{x^2-x}}+1\right)=2,$$

we recover

$$\lim_{x\to\infty}\sqrt{x^2-x}-x=-\frac{1}{2}.$$

Either approach is fine!

**(Practice #5) Evaluate**

$$\lim_{x\to\infty}(x-\ln x).$$

**SOLUTION:**

Time to be crafty! Observe that $x=\ln e^x$. Then

$$\begin{aligned}\lim_{x\to\infty}(x-\ln x) &= \lim_{x\to\infty}(\ln e^x-\ln x)\\ &= \lim_{x\to\infty}\left(\ln\left(\frac{e^x}{x}\right)\right)\\ &= \ln\left[\lim_{x\to\infty}\left(\frac{e^x}{x}\right)\right],\end{aligned}$$

since ln is continuous over a positive domain. Evaluating the inner limit, $\lim_{x\to\infty}(e^x/x)$, note that

$$\begin{aligned}\lim_{x\to\infty}e^x &= \infty\\ \lim_{x\to\infty}x &= \infty,\end{aligned}$$

so we have an indeterminate form of type $0/0$. We can therefore use L'Hospital's rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x}{x} &\overset{H}{=} \lim_{x\to\infty}\left(\frac{e^x}{1}\right)\\ &= \lim_{x\to\infty}(e^x)\\ &= \infty.\end{aligned}$$

And since $\lim_{z\to\infty}\ln(z)=\infty$, we find that

$$\lim_{x\to\infty}(x-\ln x)=\infty.$$