Induction proofs - practice!
SOLUTIONS

1. Prove that $f(n) = 6n^2 + 2n + 15$ is odd for all $n \in \mathbb{Z}^+$.

Proof by induction:

Base case: For $n = 1$, $f(1) = 4(1)^2 + 2(1) + 13 = 19$. Since 19 is odd, $f(1)$ is odd - base case proven.

Inductive hypothesis: Assume $f(n) = 4n^2 + 2n + 13$ for some $n \in \mathbb{Z}^+$. That means that $4n^2 + 2n + 13 = 2k + 1$ for some $k \in \mathbb{Z}^+$.

Inductive step: Show true for $n + 1$, that is, that $f(n+1)$ is odd.

$f(n+1) = 4(n+1)^2 + 2(n+1) + 13 = (4n^2 + 8n + 4) + (2n + 1) + 13 = (4n^2 + 2n + 13) + 8n + 14.$

But $4n^2 + 2n + 13$ is odd, and therefore can be written as $2k + 1$ for some $k \in \mathbb{Z}^+$. So

$$f(n+1) = 2k + 1 + 8n + 14 = 2(k + 4n + 7) + 1,$$

re-arranging a bit. Since $n, k \in \mathbb{Z}^+$, $k + 4n + 7 \in \mathbb{Z}^+$ and therefore, $2(k + 4n + 7) + 1$ is odd. Thus we have shown that, if $f(n)$ is odd, $f(n+1)$ must be odd, too.

Since the base case, that $f(1)$ is odd, is true, and since we have shown that if the statement is true for n, it is true for $n + 1$, we have proven that $f(n) = 6n^2 + 2n + 15$ is odd for all $n \in \mathbb{Z}^+$.

---

2. Prove that $n! > 2^n$ for n an integer greater than 4.

Proof by induction:

Base case: For $n = 4$, $n! = 4! = 24$ and $2^n = 2^4 = 16$. $24 > 16$, so $4! > 2^4$. Base case proven.

Inductive hypothesis: Assume that $n! > 2^n$ for some $n \geq 4$.

Inductive step: Show true for $n + 1$, that is, that $(n+1)! > 2^{n+1}$.

Starting with the left-hand side,

$$\begin{aligned} (n+1)! &= (n+1)n! \\ &> (n+1)2^n, \end{aligned}$$

since $n! > 2^n$ (inductive hypothesis). Now since $n \geq 4$, $n + 1 \geq 5$; we can therefore safely say that $n + 1 > 2$ and so,

$$(n+1)! \quad > \quad 2 \cdot 2^n = 2^{n+1}.$$

Thus we have shown that, if $n! > 2^n$, it follows that $(n+1)! > 2^{n+1}$.

Since the base case, that $4! > 2^4$ is true, and since we have shown that if $n! > 2^n$, it follows that $(n+1)! > 2^{n+1}$, we have proven that $n! > 2^n$ for n an integer greater than 4.

---

3. If n is a non-negative integer, show that $n^5 - n$ is divisible by 5.

Proof by induction:

Base case: For $n = 1$, $n^5 - 1 = (1)^5 - 1 = 0$, which is divisible by 5. Base case proven.

Inductive hypothesis: Assume that $n^5 - n$ is divisible by 5 for some n a non-negative integer. This means that $n^5 - n = 5k$ for some $k \in \mathbb{Z}^+$.

Inductive step: Show true for $n+1$, that is, that $(n+1)^5 - (n+1)$ is divisible by 5.

Expand the expression:

$$\begin{aligned}(n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + 1).\end{aligned}$$

Now, $n^5 - n$ is divisible by 5, so we can write it as $n^5 - n = 5k$ for some $k \in \mathbb{Z}^+$. Then

$$\begin{aligned}(n+1)^5 - (n+1) &= 5k + 5(n^4 + 2n^3 + 2n^2 + 1). \\ &= 5(k + n^4 + 2n^3 + 2n^2 + 1).\end{aligned}$$

This shows $(n+1)^5 - (n+1)$ is an integer multiple of 5, so $(n+1)^5 - (n+1)$ is divisible by 5. We have now shown that, if $n^5 - n$ is divisible by 5, it follows that $(n+1)^5 - (n+1)$ is divisible by 5 too.

The base case, that $n^5 - n$ is divisible by 5 for $n = 1$ is true, and we have shown that if $n^5 - n$ is divisible by 5, it follows that $(n+1)^5 - (n+1)$ is divisible by 5 too. Thus we have proven that if n is a non-negative integer, show that $n^5 - n$ is divisible by 5.

---

4. For each $n \in \mathbb{Z}^+$, it follows that $2^n \leq 2^{n+1} - 2^{n-1} - 1$. Show using induction.

Proof by induction:

Base case: For $n = 1$, LHS=$2^1 = 2$ and RHS=$2^{(1+1)} - 2^{(1-1)} - 1 = 2^2 - 2^0 - 1 = 2$. LHS=RHS for and therefore for $n = 1$, $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds.

Inductive hypothesis: Assume that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ for some $n \in \mathbb{Z}^+$.

Inductive step: Show true for $n+1$, that is, that $2^{n+1} \leq 2^{n+2} - 2^n - 1$.

Start with the left-hand side.

$$2^{n+1} = 2 \cdot 2^n.$$

but $2^n \leq 2^{n+1} - 2^{n-1} - 1$. Therefore,

$$\begin{aligned}2^{n+1} &\leq 2(2^{n+1} - 2^{n-1} - 1) \\ &\leq 2^{n+2} - 2^n - 2\end{aligned}$$

and then since $-2 < -1, 2^{n+2} - 2^n - 2 < 2^{n+2} - 2^n - 1$. Thus,

$$2^{n+1} \leq 2^{n+2} - 2^n - 1$$

We have shown that, if $2^n \le 2^{n+1} - 2^{n-1} - 1$ for some $n \in \mathbb{Z}^+$, it follows that $2^{n+1} \le 2^{n+2} - 2^n - 1$.

Since the base case, that $2^{n+1} \le 2^{n+2} - 2^n - 1$ for $n = 1$ is true, and since we have shown that if $2^n \le 2^{n+1} - 2^{n-1} - 1$ for some $n \in \mathbb{Z}^+$, it follows that $2^{n+1} \le 2^{n+2} - 2^n - 1$, we have proven that $2^n \le 2^{n+1} - 2^{n-1} - 1$ for each $n \in \mathbb{Z}^+$.

---

5. Prove by induction that $1+2+3+4+...+n = n(n+1)/2$ for $n \in \mathbb{Z}^+$.

Proof by induction:

Base case: For $n = 1$: LHS = 1+2+3+...+n=1. RHS = $1(1+1)/2 = 1$. Since LHS=RHS, $1+2+3+4+...+n = n(n+1)/2$ for $n = 1$.

Inductive hypothesis: Assume true for some $n \ge 1$, that is, that $1+2+3+4+...+n = n(n+1)/2$.

Inductive step: Show true for some $n+1$, that is, that $1+2+3+4+...+n+n+1 = (n+1)(n+2)/2$.

Start with the left-hand side.

$$\text{LHS}=1+2+3+4+...+n+n+1 = (1+2+3+4+...+n)+n+1.$$

But $1+2+3+4+...+n = n(n+1)/2$.

$$\begin{aligned}
\text{LHS}=1+2+3+4+...+n+n+1 &= \frac{n(n+1)}{2}+n+1 \\
&= (n+1)\left(\frac{n}{2}+1\right) \\
&= (n+1)\left(\frac{1}{2}\right)(n+2) \\
&= \frac{(n+1)(n+2)}{2} \\
&= \text{RHS.}
\end{aligned}$$

Since the base case $n = 1$ is true, and we have shown that if the statement is true for n, it is true for $n+1$, we have shown that $n \in \mathbb{Z}^+$, $1+2+3+4+...+n = n(n+1)/2$.

---

6. Prove that $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{Z}^+$.

Proof by induction:

Base case: For $n = 1$: LHS = $(1)^2 = 1$. RHS = 1^3. Since LHS=RHS, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ for $n = 1$.

Inductive hypothesis: Assume true for some $n \ge 1$, that is, that $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Inductive step: Show true for some $n+1$, that is, that $(1+2+3+\cdots+n+(n+1))^2 = 1^3+2^3+3^3+\cdots+n^3+(n+1)^3$

Start with the left-hand side.

$$\begin{aligned}\text{LHS} &= (1+2+3+\cdots+n+(n+1))^2 \\ &= (1+2+3+\cdots+n)^2+2(n+1)(1+2+3+\cdots+n)+(n+1)^2\end{aligned}$$

From the inductive hypothesis, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$. And remember that $1+2+3+\ldots+n = n(n+1)/2$ (SEE #5). Then

$$\begin{aligned}\text{LHS} &= (1^3+2^3+3^3+\cdots+n^3)+2(n+1)\left(n(n+1)/2\right)+(n+1)^2 \\ &= (1^3+2^3+3^3+\cdots+n^3)+n(n+1)^2+(n+1)^2 \\ &= (1^3+2^3+3^3+\cdots+n^3)+(n+1)(n+1)^2 \\ &= 1^3+2^3+3^3+\cdots+n^3+(n+1)^3 \\ &= \text{RHS}\end{aligned}$$

Since the base case $n=1$ is true, and we have shown that if the statement is true for n, it is true for $n+1$, we have shown that $n \in \mathbb{Z}^+$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

---

7. Prove by induction that $1+3+5+7+...+(2n-1) = n^2$ for $n \in \mathbb{Z}^+$.

Proof by induction:

Base case: For $n=1$: LHS = 1. RHS = $(1)^2 = 1$. Since LHS=RHS, $1+3+5+...+(2n-1) = n^2$ for $n=1$.

Inductive hypothesis: Assume true for some $n \geq 1$, that is, that $1+3+5+7+...+(2n-1) = n^2$.

Inductive step: Show true for some $n+1$, that is, that $1+3+5+7+...+(2n-1)+(2n+1) = (n+1)^2$.

Start with the left-hand side.

$$\begin{aligned}\text{LHS} &= 1+3+5+7+...+(2n-1)+(2n+1) \\ &= n^2+(2n+1),\end{aligned}$$

by the inductive hypothesis. But $n^2+2n+1 = (n+1)^2$. Therefore,

$$\begin{aligned}\text{LHS} &= (n+1)^2 \\ &= \text{RHS}\end{aligned}$$

$$\begin{aligned}\text{LHS}=1+2+3+4+...+n+n+1 &= \frac{n\,(n+1)}{2}+n+1 \\ &= (n+1)\left(\frac{n}{2}+1\right) \\ &= (n+1)\left(\frac{1}{2}\right)(n+2) \\ &= \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Since the base case $n = 1$ is true, and we have shown that if the statement is true for n, it is true for $n+1$, we have shown that $n \in \mathbb{Z}^+$, $1+3+5+7+...+(2n-1) = n^2$.

---

8. Show by induction that $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$ for $n \in \mathbb{Z}^+$.

Proof by induction:

Base case: For $n = 1$: LHS = $2^1 = 2$. RHS = $(2)^{1+1} - 2 = 2$. Since LHS=RHS, $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$ for $n = 1$.

Inductive hypothesis: Assume true for some $n \geq 1$, that is, that $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$.

Inductive step: Show true for some $n+1$, that is, that $2^1 + 2^2 + 2^3 + \ldots + 2^n + 2^{n+1} = 2^{n+2} - 2$.

Start with the left-hand side.

$$\begin{aligned} \text{LHS} &= (2^1 + 2^2 + 2^3 + \ldots + 2^n) + 2^{n+1} \\ &= (2^{n+1} - 2) + 2^{n+1}, \end{aligned}$$

since by the inductive hypothesis, $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$. Then

$$\begin{aligned} \text{LHS} &= 2 \cdot 2^{n+1} - 2 \\ &= 2^{n+2} - 2 \\ &= \text{RHS}. \end{aligned}$$

Since the base case $n = 1$ is true, and we have shown that if the statement is true for n, it is true for $n+1$, we have shown that $n \in \mathbb{Z}^+$, $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$.

---

9. For n an integer greater or equal to 3, show that $n^{n+1} > (n+1)^n$.

Proof by induction:

First note that the statement $n^{n+1} > (n+1)^n$ is equivalent to

$$n > \left(1 + \frac{1}{n}\right)^n, \; n \geq 3.$$

We'll prove this latter form.

Base case: For $n = 3$. LHS=3 and RHS=$(1+1/3)^3 = 64/27 < 3 = $ LHS. Base case done.

Inductive hypothesis: Assume true for some $n \geq 3$, that is, that $n > \left(1 + \frac{1}{n}\right)^n$ for some $n \geq 3$.

Inductive step: Show true for some $n+1$, that is, that $n+1 > \left(1 + \frac{1}{n+1}\right)^{n+1}$.

This time start on the right hand side.

$$\begin{aligned} \text{RHS} &= \left(1+\frac{1}{n+1}\right)^{n+1} \\ &< \left(1+\frac{1}{n}\right)^{n+1}, \\ &= \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) \\ &< n\left(1+\frac{1}{n}\right) \end{aligned}$$

since $\frac{1}{n} > \frac{1}{n+1}$ for $n \geq 1$ (and here, $n \geq 3$). Then

$$\begin{aligned} \text{RHS} &< \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) \\ &< n\left(1+\frac{1}{n}\right), \end{aligned}$$

since $\left(1+\frac{1}{n}\right)^n < n$ (inductive hypothesis. Finally, distributing the n,

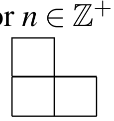
$$\begin{aligned} \text{RHS} &< n\left(1+\frac{1}{n}\right) = n+1 = \text{LHS} \\ \Rightarrow \text{LHS} &> \text{RHS} \\ \text{that is, } n+1 &> \left(1+\frac{1}{n+1}\right)^{n+1} \end{aligned}$$

Thus we have shown that, assuming $n > \left(1+\frac{1}{n}\right)^n$ is true, it follows that $\left(1+\frac{1}{n+1}\right)^{n+1}$.

And since the $n > \left(1+\frac{1}{n}\right)^n$ holds for $n = 3$, we have shown by induction that $n > \left(1+\frac{1}{n}\right)^n$ holds for $n > 3$.

Therefore, the equivalent formulation $n^{n+1} > (n+1)^n$ holds for $n \geq 3$, $n \in \mathbb{Z}$.

---

10. Prove that for $n \in \mathbb{Z}^+$, a $2^n \times 2^n$ chessboard with any one square removed can be tiled by these 3-square L-tiles: 

Proof by induction:

Base case: For $n = 1$, we have a 2×2 board with 1 square removed, which can be tiled by 1 L-tile.

Inductive hypothesis: Assume true for some $n \geq 1$, that is, that a $2^n \times 2^n$ chessboard with any one square removed can be tiled by the 3-square L-tiles.

Inductive step: Show true for some $n+1$, that is, that a $2^{n+1} \times 2^{n+1}$ chessboard with any one square removed can be tiled by the 3-square L-tiles.

Divide the $2^{n+1} \times 2^{n+1}$ board as follows, where A, B, C, D are each a $2^n \times 2^n$ board:

A	B
C	D

Without loss of generality, suppose that the one square has been removed from B. Then by the inductive hypothesis, B can be tiled. Remove the center corners of A, C, and D, such that each of their remainders can be tiled by the inductive hypothesis. Tile the remaining 3 squares in the center with a single L-tile, and we have completed the tiling.

Since the base case $n = 1$ is true, and we have shown that if the statement is true for n, it is true for $n+1$, we have shown that $n \in \mathbb{Z}^+$, a $2^n \times 2^n$ chessboard with any one square removed can be tiled by the 3-square L-tiles.

---

11. *Collatz conjecture:* Take any natural number n. If n is even, divide it by 2 to get $n/2$. If n is odd, multiply it by 3 and add 1 to obtain $3n+1$. Repeat the process indefinitely. The Collatz conjecture states that, no matter what number you start with, you will always eventually reach 1.

Here's a hint.

Proof: There isn't a known proof of the Collatz conjecture! (that's why it's a *conjecture*).
