Newton's law of cooling example

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution: In general, we want to solve

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_S) \\ T(0) &= T_0 \end{aligned}$$

Let $y = T - T_S$. Then $y' = T'$, $y(0) = T(0) - T_S = T_0 - T_S$, and

$$\begin{aligned} \frac{dy}{dt} &= ky \\ y(0) &= T_0 - T_S. \end{aligned}$$

As we saw in class, this equation has solution

$$y(t) = (T_0 - T_S)\, e^{kt}.$$

Now since $y(t) = T - T_S$ we're left with

$$T(t) = (T_0 - T_S)\, e^{kt} + T_S$$

as our temperature at time t, with t in minutes.

For OUR problem, $T_S = 44°\text{F}$ and $T_0 = 72°\text{F}$, so that

$$T(t) = (28°\text{F})\, e^{kt} + 44°\text{F}.$$

k is unknown; to solve for k use $T(30) = 61°\text{F}$:

$$61°\text{F} = (28°\text{F})\, e^{30k} + 44°\text{F}.$$

Solving for k we find $k = -(\ln(28) - \ln(17))/30 \approx -0.017$. Therefore our model for the pop cooling is

$$T(t) = (28°\text{F})\, e^{-0.017t} + 44°\text{F}.$$

(a) The temperature of soda after another half hour is $T(60) = (28°\text{F})\, e^{-0.017(60)} + 44°\text{F} = 54°\text{F}$ to the nearest degree.

(b) Want time $\tilde{t}$ such that $T(\tilde{t}) = 50°\text{F}$. So $50°\text{F} = (28°\text{F})\, e^{-0.017\tilde{t}} + 44°\text{F}$. Solving for $\tilde{t}$ we find that in takes 93 minutes, to the nearest minute, to cool to 50°F.

Note as the temperature of the soda approaches T_S, the rate of change gets slower (decay slows down). You can convince yourself by plotting $T(t)$ and/or by remembering the original differential equation: $T'(t) = k(T - T_S)$. For T close to T_S, $T - T_S$ is small, so $T'(t) = k(T - T_S)$ gets small, too.