# Math 141H-01

# PRACTICE Midterm Exam 1

**February X, 2015**

Name:________________________________________

**Instructions:** Clearly answer each of the questions. Box your answers. Partial credit will be awarded based on the clarity and correctness of your explanation of each solution. Use the backs of pages for your scratch work. Make sure you have all 11 pages.

*Trigonometric identities and useful integration formulas are given on pages 10-11.*

**1.** (X points) Inverse Functions.
(a) Does $f(x) = x^2 - 1$ have an inverse? Why or why not?
**Solution:** $f(x)$ does not have an inverse, it's not 1-to-1 - it doesn't pass the horizontal line test. E.g. $f(-1) = f(1) = 0$.

(b) Given $g(x) = x^3 - 8$, what is $g^{-1}(-8)$?
**Solution:** $g^{-1}(-8) = 0$.

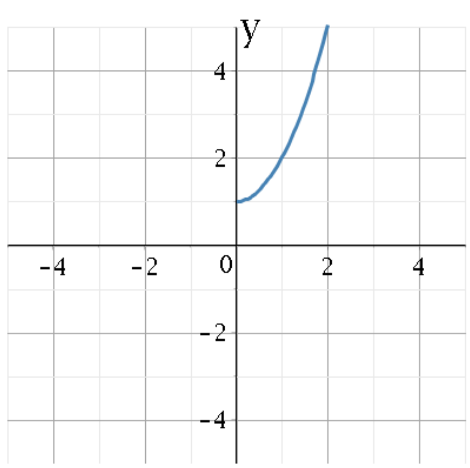
(c) Given $h^{-1}(x) = \tan^{-1}(2x+1)$, what is $h(\pi/4)$?
**Solution:** $h(\pi/4) = 0$.

(d) Let $f$ be a function such that

$$f(-4) = 5, \quad f(1) = 2, \quad f(2) = 1, \quad f(3) = -4, \quad f(5) = -2$$
$$f'(-4) = 3, \quad f'(-2) = 1, \quad f'(2) = \frac{1}{4}, \quad f'(3) = -\frac{1}{4}, \quad f'(5) = 2$$

Find $(f^{-1})'(-2)$ and $(f^{-1})'(5)$.
**Solution:** $(f^{-1})'(-2) = 1/2$ and $(f^{-1})'(5) = 1/3$.

(e) $w(x) = \sqrt{x-1}$. Sketch $w^{-1}(x)$



**2.** (X points)

(a) What is the domain of $y = \ln(x^2 - 1)$?
**Answer:** $|x| > 1$

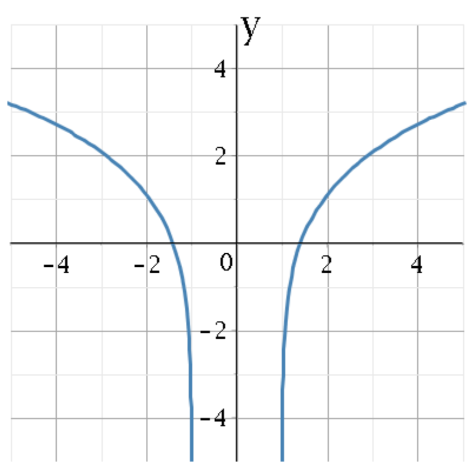
(b) What is the range of $y = \tan^{-1}(x^2)$?
**Answer:** $0 \leq y < \pi/2$

(c) Determine the $x$-intercept(s) of the graph $y = e^{x^2-1} - 1$.
**Answer:** $x = -1$ and $1$

(d) Determine the $y$-intercept(s) of the graph $y = 2^{1/(x+1)} - 4$.
**Answer:** $y = -2$

(e) Compute $\lim_{x\to\infty}(\ln(1+2x) - \ln(1+3x))$.
**Answer:** $\lim_{x\to\infty}(\ln(1+2x) - \ln(1+3x)) = \ln(2/3)$

(f) Sketch $y = \ln(x^2 - 1)$.



**3.** (X points) Find the derivative of the given function, evaluated at the given point. Simplify completely (no inverse trig functions in your answer!).

(a) $y = \tan(\sin^{-1}(t))$, $t = 0$.

**Answer:**

$$\frac{dy}{dt} = \frac{1}{(1-t^2)^{3/2}}; \quad \frac{dy}{dt}|_{t=0} = 1$$

(b) $y = (\ln x)^{e^x}$, $x = e$.

**Answer:**

$$\frac{dy}{dx} = e^x (\ln x)^{e^x} \left( \ln(\ln x)) + \frac{e^x}{x \ln x} \right); \quad \frac{dy}{dx}|_{x=e} = e^{e-1}$$

**4.** (X points) Determine the correct partial fractions expansions. DO NOT DETERMINE THE CONSTANTS AND DO NOT INTEGRATE.

(a) To integrate

$$\int \frac{x}{x^2 - 7x + 12}\, dx,$$

which partial fractions expansion should you use? DO NOT DETERMINE THE CONSTANTS AND DO NOT INTEGRATE. **Answer:** $x^2 - 7x + 12 = (x-3)(x-4)$, so you would use

$$\frac{x}{x^2 - 7x + 12} = \frac{A}{x-3} + \frac{B}{x-4}$$

(b) To integrate

$$\int \frac{x}{(x^2-1)^2(x^2-3x+12)^2}\, dx,$$

which partial fractions expansion should you use? DO NOT DETERMINE THE CONSTANTS AND DO NOT INTEGRATE. **Answer:** $(x^2-1)^2(x^2-3x+12)^2 = (x-1)^2(x+1)^2(x^2-3x+12)^2$, noting that $x^2 - 3x + 12$ is irreducible (the discriminant $b^2 - 4ac$ is negative, the expression can't be factored). You would therefore use

$$\frac{x}{(x^2-1)^2(x^2-3x+12)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2-3x+12} + \frac{Gx+H}{(x^2-3x+12)^2}$$

**5.** (X points) Integrate by parts

$$\int_0^1 x\tan^{-1} x\,dx$$

**Answer:** Using LIATE... "I" for inverse trig.

$$\begin{aligned} u &= \tan^{-1} x & \qquad dv &= x\,dx \\ du &= dx/(1+x^2) & \qquad v &= x^2/2. \end{aligned}$$

Then

$$\int_0^1 x\tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx.$$

In new integrand, degree of numerator = degree of denominator, so before we can use partial fractions expansions (which we won't actually have to do, as you'll see), we have to perform the polynomial long division:

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{1+x^2}.$$

Our integral becomes

$$\begin{aligned} \int_0^1 x\tan^{-1} x\,dx &= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)\,dx \\ &= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C \end{aligned}$$

Therefore the final answer is

$$\int_0^1 x\tan^{-1} x\,dx = \frac{\tan^{-1} x}{2}\left(x^2+1\right) - \frac{x}{2} + C$$

**6.** (X points) Integrate using a trigonometric substitution. Simplify as much as you can.

$$\int \frac{x^3}{\sqrt{9+4x^2}}\,dx$$

**Answer:** Let $x = (3/2)\tan\theta$. Then $dx = (3/2)\sec^2\theta\,d\theta$, $x^3 = 27\tan^3\theta/8$, and $\sqrt{9+4x^2} = \sqrt{9+9\tan^2\theta} = 3\sec\theta$. Thus,

$$\begin{aligned}\int \frac{x^3}{\sqrt{9+4x^2}}\,dx &= \int \frac{((27/8)\tan^3\theta)}{(3\sec\theta)}((3/2)\sec^2\theta)\,d\theta \\ &= \frac{27}{16}\int \tan^3\theta\sec\theta d\theta\end{aligned}$$

Now we use our trig integrals techniques. Use the identity $\tan^2\theta = \sec^2\theta - 1$ and then the u-substitution $u = \sec\theta$, $du = \sec\theta\tan\theta d\theta$ to obtain

$$\begin{aligned}\int \frac{x^3}{\sqrt{9+4x^2}}\,dx &= \frac{27}{16}\int u^2 - 1\,du \\ &= \frac{27}{16}\left(\frac{u^3}{3} - u\right) + C \\ &= \frac{9}{16}\left(\sec^3\theta - 3\sec\theta\right) + C,\end{aligned}$$

since $u = \sec\theta$. But $\theta = \tan^{-1}(2x/3)$! Using triangles we recover $\sec(\theta) = \sqrt{9+4x^2}/3$. Therefore our final answer is

$$\int \frac{x^3}{\sqrt{9+4x^2}}\,dx = \frac{1}{24}(2x^2-9)\sqrt{9+4x^2} + C$$

**7.** (X points) Integrate using whatever method you wish.

(a) $\displaystyle\int_0^{\pi/4} \cos^2\theta\,\sin^2\theta\,d\theta$

**Answer:** Use the identities $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$ and $\sin^2(2\theta) = (1-\cos(4\theta))/2$ to obtain

$$\int \cos^2\theta\,\sin^2\theta\,d\theta = \frac{\theta}{8} - \frac{1}{32}\sin(4\theta) + C$$

Then evaluating the definite integral,

$$\int_0^{\pi/4} \cos^2\theta\,\sin^2\theta\,d\theta = \frac{\pi}{32}$$

(b) $\displaystyle\int \sin(t)\ln(\cos^2(t))\,dt$

**Answer:** First use properties of logarithms to write the integral as

$$\int \sin(t)\ln(\cos^2(t))\,dt = 2\int \sin(t)\ln(\cos(t))\,dt$$

Then use the substitution $w = \cos(t)$, so $dw = -\sin(t)$ and we're left with

$$\int \sin(t)\ln(\cos^2(t))\,dt = -2\int \ln w\,dw.$$

Integrate this by parts with $u = \ln w$ and $dv = dx$ (as in class) to find

$$\int \sin(t)\ln(\cos^2(t))\,dt = 2\,(w - w\ln w) + C$$

But $w = \cos(t)$ so our final answer is

$$\int \sin(t)\ln(\cos^2(t))\,dt = 2\cos t\,(1-\ln\cos t) + C$$

(c) $\displaystyle\int \frac{x}{\sqrt{x-x^2}}\,dx$

**Answer:** $-x^2$ in $\sqrt{\cdot}$ in the denominator suggests an arcsine function. Re-write the denominator by completing the square, $x-x^2 = 1/4-(x-1/2)^2$ which simplifies to $1/4\left(1-(2x-1)^2\right)$. Our integral becomes

$$\int \frac{x}{\sqrt{x-x^2}}\,dx = \int \frac{2x}{\sqrt{1-(2x-1)^2}}\,dx$$

For convenience re-write as

$$\int \frac{x}{\sqrt{x-x^2}}\,dx = \int \frac{2x-1}{\sqrt{1-(2x-1)^2}}\,dx + \int \frac{1}{\sqrt{1-(2x-1)^2}}\,dx.$$

To evaluate, in the first integral use the $u$-sub $u = 1-(2x-1)^2$, in the second use the $u$-sub $u = 2x-1$, to recover (after some simplification)

$$\int \frac{x}{\sqrt{x-x^2}}\,dx = -\sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$$

(d) $\displaystyle\int_2^3 \frac{x^2}{x^2-1}\,dx$

**Answer:** Rational function with degree of the numerator $\geq$ degree of the denominator so we first reduce via polynomial long division:

$$\frac{x^2}{1-x^2} = 1 + \frac{1}{x^2-1}.$$

To integrate, we'll have to expand the second term via partial fractions expansion,

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

Now integrate.

$$\begin{aligned}
\int_2^3 \frac{x^2}{x^2-1}\,dx &= \int_3^2 \left(1 + \frac{1}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)\right)\,dx \\
&= x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\Big|_2^3 \\
&= 1 + \frac{1}{2}\ln(3) - \frac{1}{2}\ln(2)
\end{aligned}$$

Our final answer is therefore

$$\int_2^3 \frac{x^2}{x^2-1}\,dx = 1 + \frac{1}{2}\ln(3) - \frac{1}{2}\ln(2)$$

or $1+\ln(\sqrt{3/2})$ if you want to get fancy.