CHAPTER 10

Mathematical Induction

This chapter explains a powerful proof technique called **mathematical induction** (or just **induction** for short). To motivate the discussion, let's first examine the kinds of statements that induction is used to prove. Consider the following statement.

Conjecture. The sum of the first n odd natural numbers equals n^2.

The following table illustrates what this conjecture says. Each row is headed by a natural number n, followed by the sum of the first n odd natural numbers, followed by n^2.

n	sum of the first n odd natural numbers	n^2
1	$1 =$	1
2	$1+3 =$	4
3	$1+3+5 =$	9
4	$1+3+5+7 =$	16
5	$1+3+5+7+9 =$	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+9+11+\cdots+(2n-1) =$	n^2
$\vdots$	$\vdots$	$\vdots$

Note that in the first five lines of the table, the sum of the first n odd numbers really does add up to n^2. Notice also that these first five lines indicate that the nth odd natural number (the last number in each sum) is $2n-1$. (For instance, when $n = 2$, the second odd natural number is $2\cdot2-1 = 3$; when $n = 3$, the third odd natural number is $2\cdot3-1 = 5$, etc.)

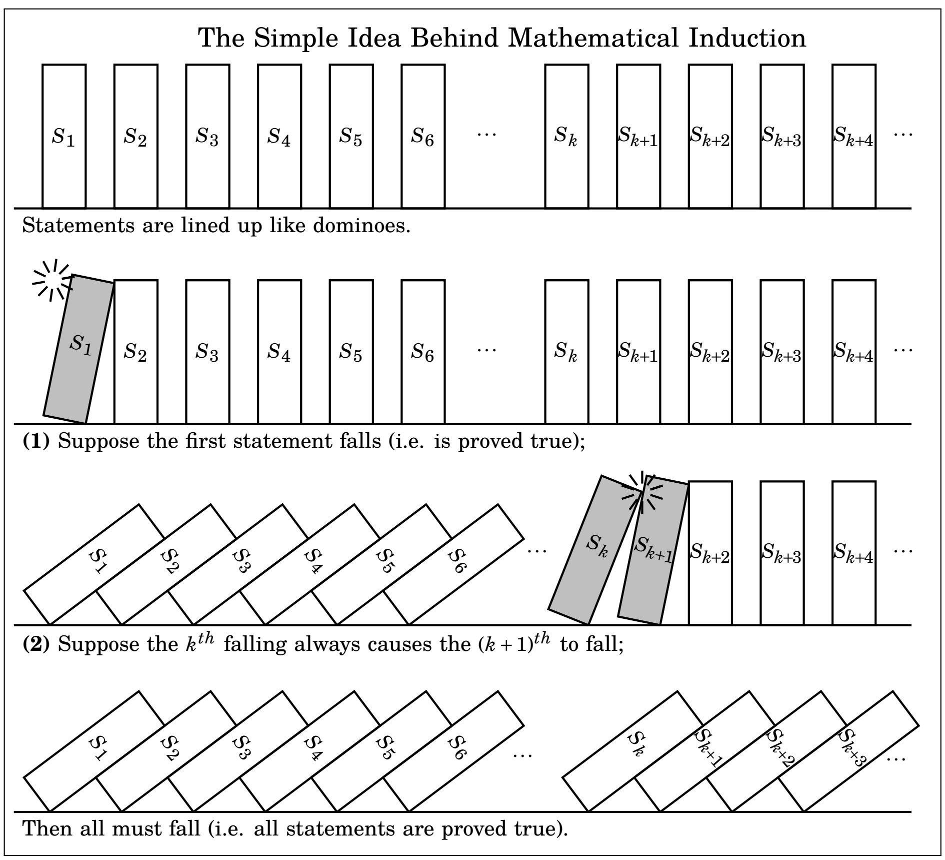
The table raises a question. Does the sum $1+3+5+7+\cdots+(2n-1)$ really always equal n^2? In other words, is the conjecture true?

Let's rephrase this as follows. For each natural number n (i.e., for each line of the table), we have a statement S_n, as follows:

$$\begin{aligned}&S_1 : 1 = 1^2\\&S_2 : 1+3 = 2^2\\&S_3 : 1+3+5 = 3^2\\&\vdots\\&S_n : 1+3+5+7+\cdots+(2n-1) = n^2\\&\vdots\end{aligned}$$

Our question is: Are all of these statements true?

Mathematical induction is designed to answer just this kind of question. It is used when we have a set of statements $S_1, S_2, S_3, \ldots, S_n, \ldots$, and we need to prove that they are all true. The method is really quite simple. To visualize it, think of the statements as dominoes, lined up in a row. Imagine you can prove the first statement S_1, and symbolize this as domino S_1 being knocked down. Additionally, imagine that you can prove that any statement S_k being true (falling) forces the next statement S_{k+1} to be true (to fall). Then S_1 falls, and knocks down S_2. Next S_2 falls and knocks down S_3, then S_3 knocks down S_4, and so on. The inescapable conclusion is that all the statements are knocked down (proved true).



This picture gives our outline for *proof by mathematical induction.*

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Induction)
(1) Prove that the first statement S_1 is true.
(2) Given any integer $k \geq 1$, prove that the statement $S_k \Rightarrow S_{k+1}$ is true.
It follows by mathematical induction that every S_n is true. ■

In this setup, the first step (1) is called the **basis step**. Because S_1 is usually a very simple statement, the basis step is often quite easy to do. The second step (2) is called the **inductive step**. In the inductive step direct proof is most often used to prove $S_k \Rightarrow S_{k+1}$, so this step is usually carried out by assuming S_k is true and showing this forces S_{k+1} to be true. The assumption that S_k is true is called the **inductive hypothesis**.

Now let's apply this technique to our original conjecture that the sum of the first n odd natural numbers equals n^2. Our goal is to show that for each $n \in \mathbb{N}$, the statement $S_n : 1+3+5+7+\cdots+(2n-1) = n^2$ is true. Before getting started, observe that S_k is obtained from S_n by plugging k in for n. Thus S_k is the statement $S_k : 1+3+5+7+\cdots+(2k-1) = k^2$. Also, we get S_{k+1} by plugging in $k+1$ for n, so that $S_{k+1} : 1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$.

Proposition If $n \in \mathbb{N}$, then $1+3+5+7+\cdots+(2n-1) = n^2$.

Proof. We will prove this with mathematical induction.

(1) Observe that if $n = 1$, this statement is $1 = 1^2$, which is obviously true.

(2) We must now prove $S_k \Rightarrow S_{k+1}$ for any $k \geq 1$. That is, we must show that if $1+3+5+7+\cdots+(2k-1) = k^2$, then $1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$. We use direct proof. Suppose $1+3+5+7+\cdots+(2k-1) = k^2$. Then

$$\begin{aligned}
1+3+5+7+\cdots\cdots\cdots\cdots\cdots+(2(k+1)-1) &= \\
1+3+5+7+\cdots+(2k-1)+(2(k+1)-1) &= \\
\big(1+3+5+7+\cdots+(2k-1)\big)+(2(k+1)-1) &= \\
k^2 \quad +(2(k+1)-1) &= k^2+2k+1 \\
&= (k+1)^2.
\end{aligned}$$

Thus $1+3+5+7+\cdots+(2(k+1)-1) = (k+1)^2$. This proves that $S_k \Rightarrow S_{k+1}$.

It follows by induction that $1+3+5+7+\cdots+(2n-1) = n^2$ for every $n \in \mathbb{N}$. ■

In induction proofs it is usually the case that the first statement S_1 is indexed by the natural number 1, but this need not always be so. Depending on the problem, the first statement could be S_0, or S_m for any other integer m. In the next example the statements are $S_0, S_1, S_2, S_3, \ldots$ The same outline is used, except that the basis step verifies S_0, not S_1.

Proposition If n is a non-negative integer, then $5 \mid (n^5 - n)$.

Proof. We will prove this with mathematical induction. Observe that the first non-negative integer is 0, so the basis step involves $n = 0$.

(1) If $n = 0$, this statement is $5 \mid (0^5 - 0)$ or $5 \mid 0$, which is obviously true.

(2) Let $k \geq 0$. We need to prove that if $5 \mid (k^5 - k)$, then $5 \mid ((k+1)^5 - (k+1))$. We use direct proof. Suppose $5 \mid (k^5 - k)$. Thus $k^5 - k = 5a$ for some $a \in \mathbb{Z}$. Observe that

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5a + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5(a + k^4 + 2k^3 + 2k^2 + k).
\end{aligned}$$

This shows $(k+1)^5 - (k+1)$ is an integer multiple of 5, so $5 \mid ((k+1)^5 - (k+1))$. We have now shown that $5 \mid (k^5 - k)$ implies $5 \mid ((k+1)^5 - (k+1))$.

It follows by induction that $5 \mid (n^5 - n)$ for all non-negative integers n. ■

As noted, induction is used to prove statements of the form $\forall n \in \mathbb{N}, S_n$. But notice the outline does *not* work for statements of form $\forall n \in \mathbb{Z}, S_n$ (where n is in $\mathbb{Z}$, not $\mathbb{N}$). The reason is that if you are trying to prove $\forall n \in \mathbb{Z}, S_n$ by induction, and you've shown S_1 is true and $S_k \Rightarrow S_{k+1}$, then it only follows from this that S_n is true for $n \geq 1$. You haven't proved that any of the statements $S_0, S_{-1}, S_{-2}, \ldots$ are true. If you ever want to prove $\forall n \in \mathbb{Z}, S_n$ by induction, you have to show that some S_a is true and $S_k \Rightarrow S_{k+1}$ **and** $S_k \Rightarrow S_{k-1}$.

Unfortunately, the term *mathematical induction* is sometimes confused with *inductive reasoning*, that is, the process of reaching the conclusion that something is likely to be true based on prior observations of similar circumstances. Please note that that mathematical induction, as introduced here, is a rigorous method that proves statements with absolute certainty.