Homework 5

due October 14, 2015

Instructions: For all problems upload your code to the Homework #5 CODE DROPBOX on Angel. Unless otherwise stated, answers should be given to six digits after the decimal point. For your reference, the last page includes a list of our numerical integral approximation rules.

1. The **Matlab** function Hw5Num1f.m estimates the integral

$$\int_a^b \sin(x^2)\,dx$$

using the left-endpoint approximation L_N for arbitrary number of subintervals N, lower bound a, and upper bound b (i.e. N, a, b are **inputs** to the **Matlab function** Hw5Num1f(a,b,N) that integrates $\int_a^b \sin(x^2)\,dx$). Use the code provided to approximate each of the following integrals. **Note:** To call function, make sure your working directory contains "Hw5Num1f.m." Then type

```
>> Hw5Num1f(-1,1,100)
```

at the **Matlab** command line, for example, to get the answer for (a) below.

(a) $\int_{-1}^{1} \sin(x^2)\,dx$ with $N = 10$.

(b) $\int_{-1}^{1} \sin(x^2)\,dx$ with $N = 100$.

(c) $\int_{10}^{11} \sin(x^2)\,dx$ with $N = 1000$.

(d) $\int_{10}^{11} \sin(x^2)\,dx$ with $N = 10000$.

(e) The code provided in Hw5Num1f.m explicitly spells out all the steps the computer takes, and so is easiest to understand and modify, but is rather long and slow. A faster and shorter *vectorized* version of the function is found in Hw5Num1v.m (take a look). Repeat (a)-(d) using

```
>> Hw5Num1v(-1,1,100)
```

for (a), for example, to see that you get the same answers, even though the programs look very different.

For the following problems, you can use either "Hw5Num1v.m" or "Hw5Num1f.m", depending on which style you prefer.

2. Modify the function from Problem 1 to estimate the integral

$$\int_a^b \frac{e^x}{x}\,dx$$

Call the new function "Hw5Num2", with file name "Hw5Num2.m". Then answer the following.

   (a) Approximate each of the following for $N = 1000$

   $$\int_1^2 \frac{e^x}{x}\,dx, \quad \int_2^3 \frac{e^x}{x}\,dx, \quad \int_1^3 \frac{e^x}{x}\,dx,$$

   (b) Is the actual values of the integral greater or less than your approximations? Explain your answer.

   (c) Why do you get different numbers for $\int_1^2 \frac{e^x}{x}\,dx + \int_2^3 \frac{e^x}{x}\,dx$ and $\int_1^3 \frac{e^x}{x}\,dx$?

   (d) How would you eliminate that difference?

3. Modify the function from Problem 2 so that it uses the **right**-endpoint approximation R_N, rather than L_N. Call the new function "Hw5Num3", with file name "Hw5Num3.m".

   (a) Calculate R_{1000} for the integral

   $$\int_1^3 \frac{e^x}{x}\,dx$$

   (b) Is the actual value of the integral greater or less than R_{1000}? Explain your answer.

4. Write a **Matlab** function to estimate

$$\int_a^b \frac{dx}{\ln x}$$

using the **midpoint** approximation M_N. Call the new function "Hw5Num4", with file name "Hw5Num4.m". Note that **Matlab** calls $\ln(x)$, "$\log(x)$". Then approximate each of the following using $N = 100$.

$$\text{(a)} \int_1^2 \frac{dx}{\ln x}, \quad \text{(b)} \int_2^3 \frac{dx}{\ln x},$$

(c) For (b), increase N to $N = 1000, 5000, 10000$. What happens? Why?
(d) For (a), increase N to $N = 1000, 5000, 10000$. What happens? Why?

5. Write a new function "Hw5Num5", with file name "Hw5Num5.m", to estimate

$$\int_a^b \cos(e^x)\,dx$$

using the **trapezoidal** approximation T_N.

   (a) Calculate T_N when $N = 100$, $a = 0$ and $b = 1$.

   (b) Increase N until you're convinced the first 6 digits are correct. What are those digits? What N did you need?

6. Write a new function "Hw5Num6", with file name "Hw5Num6.m", to estimate

$$\int_0^2 \frac{\sin x}{x}\, dx$$

using **Simpson's** approximation S_N. Notice that the integrand is undefined at $x = 0$, but $\lim_{x\to 0} \sin x/x = 1$, so the integral does exist.

   (a) How did you handle the removable singularity at $x = 0$ algorithmically?

   (b) Calculate S_N when $N = 10$, $a = 0$ and $b = 2$.

   (c) Then increase N until you're convinced the first 6 digits are correct. What are those digits? What N did you need?

7. **Estimating errors.**

   We saw in lecture that for the trapezoidal rule, the error E_T is typically proportional to the number of subintervals N

$$E_T = CN^{-2},$$

   where C is a constant.

   Use the exact value of the integral

$$\int_1^e \frac{1}{x}\, dx = 1$$

   to verify the error of the trapezoidal rule.

   To accomplish this task:

   (a) Modify the function provided to integrate

$$\int_1^e \frac{1}{x}\, dx$$

   using the **Trapezoidal** approximation. Call the new function "Hw5Num7", with file name "Hw5Num7.m". Note that Matlab uses e=exp(1); "exp" is the exponential function, e^x =exp(x), with numerical error $O(10^{-16})$.

   (b) Numerically integrate $\int_1^e \frac{1}{x}\, dx$ using $N = 32$, 64, 128, 256, 512, and 1024. Calculate the error for each. That is, for $\int_1^e \frac{1}{x}\, dx \approx T_N$ calculated with N sub-intervals,

$$E_T^{(N)} = |1 - T_N|,$$

   since 1 is the exact answer. Make a table of your results, like the tables on page 533 of your texbook. Use 8 digits after the decimal point.

   (c) Plot your results with $E_T^{(N)}$ on the $y-$axis and N on the $x-$axis. Consider using alternative scales for $x-$ or $y-$axes (e.g. semilog or log log scalings).

(d) Infer from the plot (using your skills with fitting lines to data) what the relationship between the error E_T and the number of subintervals N for

$$\int_1^e \frac{1}{x}\,dx.$$

Turn in your table, plot, and calculations of error from the fit.

8. **Estimating errors**

We saw in lecture that for the Simpson's rule, the error E_S is typically proportional to the number of subintervals N

$$E_S = CN^{-4},$$

where C is a constant.

Use the exact value of the integral

$$\int_1^e \frac{1}{x}\,dx = 1$$

to verify the error of the trapezoidal rule.

To accomplish this task:

(a) Modify the function provided to integrate

$$\int_1^e \frac{1}{x}\,dx$$

using **Simpson's** approximation. Call the new function "Hw5Num8", with file name "Hw5Num8.m". Note that Matlab uses e=exp(1); "exp" is the exponential function, e^x =exp(x), with numerical error $O(10^{-16})$.

(b) Numerically integrate $\int_1^e \frac{1}{x}\,dx$ using $N = 8$, 16, 32, 64, 128, 256. Calculate the error for each. That is, for $\int_1^e \frac{1}{x}\,dx \approx S_N$ calculated with N sub-intervals,

$$E_S^{(N)} = |1 - S_N|,$$

since 1 is the exact answer. Make a table of your results, like the tables on page 533 of your texbook. Use 12 digits after the decimal point.

(c) Plot your results with $E_S^{(N)}$ on the $y-$axis vs N on the $x-$axis. Consider using alternative scales for $x-$ or $y-$axes (e.g. semilog or log log scalings).

(d) Infer from the plot (using your skills with fitting lines to data) what the relationship between the error E_S and the number of subintervals N for

$$\int_1^e \frac{1}{x}\,dx.$$

Turn in your table, plot, and calculations of error from the fit.

9. **Not all functions are created equal.**

The natural impulse is to use the "best" approximation to numerically integrate. Out of the approximations we've discussed, that means Simpson's rule! But there *are* exceptional cases where an approximation method does better or worse than we expect, so understanding the strengths and weaknesses of each method is important too. This problem and the next provide illustrations.

Consider the integral

$$\int_1^3 |x^2 - 2|\, dx,$$

where $|\cdot|$ denotes absolute value. The exact value of this integral is $\int_1^3 |x^2-2|\,dx = 4(2\sqrt{2}+1)/3$. Using the approach from #7 we find the relationship between error and the number of sub-intervals N for the **Trapezoidal rule**, using $N = 10, 50, 100, 500, 1000$, is $E_T = CN^{-1.9}$, close to its expected $E_T = CN^{-2}$.

(a) Estimate the error proportionality with N (i.e., $E_T = CN^{-\alpha}$; determine α) when using **Simpson's rule** to integrate $\int_1^3 |x^2-2|\,dx$.
To accomplish this task:

i. Modify the function provided to integrate

$$\int_1^3 |x^2 - 2|\, dx$$

using **Simpson's** approximation. Call the new function "Hw5Num9", with file name "Hw5Num9.m".

ii. Numerically integrate $\int_1^3 |x^2-2|\,dx$ using $N = 10, 50, 100, 500, 1000$. Calculate the error for each. That is, for $\int_1^3 |x^2-2|\,dx \approx S_N$ calculated with N sub-intervals,

$$E_S^{(N)} = \left|\frac{4(2\sqrt{2}+1)}{3} - S_N\right|,$$

since $4(2\sqrt{2}+1)/3$ is the exact answer. Make a table of your results, like the tables on page 533 of your texbook.

iii. Plot your results with $E_S^{(N)}$ on the $y-$axis and N on the $x-$axis. Consider using alternative scales for $x-$ or $y-$axes (e.g. semilog or log log scalings). **NOTE:** The error won't be as nicely behaved as in 7 or 8.

iv. Infer from the plot (using your skills with fitting lines to data) what the relationship between the error E_S and the number of subintervals N for

$$\int_1^3 |x^2 - 2|\, dx.$$

Turn in your plot, table, and calculations of error from the fit.

(b) For $\int_1^3 |x^2-2|\,dx$, is Simpson's rule better or worse than the trapezoidal rule? Why do you think that may be? (Hint: Plot the integrand and consider the derivation of Simpson's rule).

10. The perimeter of an ellipse with major and minor axes $1/\pi$ and $0.6/\pi$, respectively, has perimeter

$$P = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-0.36\sin^2\theta}\,d\theta.$$

The exact solution is

$$P = 0.90277992777219...$$

(the integral is related to the complete elliptic function of the second kind, which is well-studied).

(a) Estimate the error proportionality with N when using the **Trapezoidal rule** to calculate P. To accomplish this task:

i. Modify the function provided to integrate

$$P = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-0.36\sin^2\theta}\,d\theta.$$

using the **Trapezoidal** approximation. Call the new function "Hw5Num10", with file name "Hw5Num10.m".

ii. Numerically integrate P using $N = 4$, 8, 12, 16, 20. Calculate the error for each. That is, for $P = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-0.36\sin^2\theta}\,d\theta \approx T_N$ calculated with N sub-intervals,

$$E_T^{(N)} = |0.90277992777219 - T_N|,$$

since 0.90277992777219... is the exact answer. Make a table of your results, like the tables on page 533 of your texbook. Use 12 digits after the decimal point.

iii. Plot your results with $E_T^{(N)}$ on the $y-$axis and N on the $x-$axis. Consider using alternative scales for $x-$ or $y-$axes (hint: semilog scaling).

iv. Infer from the plot (using your skills with fitting lines to data) what the relationship between the error E_T and the number of subintervals N for

$$P = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-0.36\sin^2\theta}\,d\theta.$$

Turn in your table, plot and calculations of error from the fit.

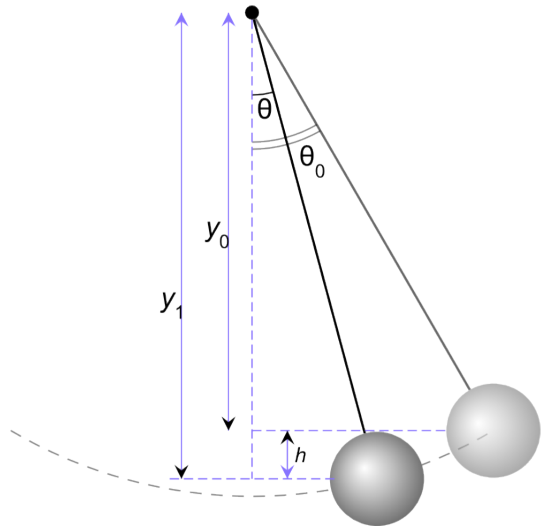
(b) You'll find the error is EXPONENTIAL, i.e. $E_T = C(a)^{-N}$, where a is a number your aim is to compute. Using Simpson's rule we find the error is $E_S \approx C(2)^{-N}$. Is the trapezoidal rule better or worse?

NOTES:

(1) Trapezoidal rule often converges exponentially when the integrand is oscillatory. (Trefethen & Weideman, SIAM Review 56(2014): 385-458).

(2) The phenomenon of exponential convergence was first noted by Poisson in the 1820s using the integral P.

11. Now, we can study one of the oldest stories of modern calculus. Consider the simple pendulum.



We're going to investigate the period of this pendulum, which we'll calculate from the angular velocity $\frac{d\theta}{dt}$. But to get there, we're going to need a little classical mechanics from physics.

By conservation of energy, any object falling a vertical distance h would acquire kinetic energy equal to the potential energy lost in the fall: gravitational potential energy is converted into kinetic energy. As the pendulum swings from angle θ_0 to angle θ the change in potential energy is given by

$$\Delta U = mgh$$

(where g is the gravitational acceleration and m the mass). The kinetic energy of a moving object is given by

$$K = \frac{1}{2}mv^2.$$

(where v is the velocity). We assume that no energy is lost. Therefore, $\Delta U = K$ implies that $mgh = \frac{1}{2}mv^2$, so that

$$v = \sqrt{2gh}. \tag{1}$$

Recall from geometry that the arclength s of a circle spanned by angle θ is $s = \ell\theta$, where ℓ is the radius. Then the velocity of the pendulum v is given by $v = \frac{ds}{dt} = \ell\frac{d\theta}{dt}$. The height h can be related to angles θ and θ_0 as $h = \ell(\cos\theta - \cos\theta_0)$, obtained by taking the difference between $y_1 = \ell\cos\theta$ and $y_0 = \ell\cos\theta_0$ in the schematic. Therefore, after some simplification, equation (1) becomes

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_0)},$$

the angular velocity. Inverting we find

$$\frac{dt}{d\theta} = \sqrt{\frac{\ell}{2g}}\frac{1}{\sqrt{(\cos\theta - \cos\theta_0)}}, \tag{2}$$

where we now consider time as a function of the angle θ.

We'll integrate (2) to compute the pendulum's period.

Over 1 complete period, the pendulum angle θ starts at $\theta = \theta_0$, passes through $\theta = 0$, goes up to $\theta = -\theta_0$, passes back through $\theta = 0$ before ending its period back at $\theta = \theta_0$:

$$T = 4\sqrt{\frac{\ell}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}, \tag{3}$$

(the period is 4×the duration of 1/4 of its cycle).

Notice that at the endpoint $\theta = \theta_0$, the integrand is singular (zero denominator). Unlike in #4 though, the area under the curve is finite. We'd have to use midpoint rule to avoid the endpoints. To use a better numerical integration method, we first need to work with the integral to remove the discontinuity at the endpoint. Starting with

$$T = 4\sqrt{\frac{\ell}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}},$$

replace $\cos\theta$ with $1 - \sin^2(\theta/2)$ and $\cos\theta_0$ with $1 - \sin^2(\theta_0/2)$ to obtain

$$T = 2\sqrt{\frac{\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0) - \sin^2(\theta)}}.$$

Now make the substitution $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$ and $\frac{1}{2}\cos(\theta/2)\,d\theta = \sin(\theta_0/2)\sin\phi\,d\phi$. When $\theta = 0$, $\phi = 0$; when $\theta = \theta_0$, $\phi = \pi/2$! The integral becomes

$$T = 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2(\phi)}}. \tag{4}$$

No discontinuities here! This is the complete elliptic integral of the first kind.

Equation (4) for the period is what we'll integrate.

(a) Plot the period T as a function of start angle θ_0. Assume gravitational acceleration g =9.8 m/s^2 and that the length of the pendulum $\ell = 0.5$m. You may use either the midpoint rule, trapezoidal rule, or Simpson's rule to perform the calculation.

(b) Often we use the "small-angle approximation": for θ_0 small, the period is $T_{\text{approx}} \approx 2\pi\sqrt{\ell/g}$. Now, say you want to design a grandfather clock. Typically $\theta_0 = \pi/30$ for a grandfather clock, which is pretty small. You therefore use the small-angle approximation to calculate the the length ℓ so that the period is approximately 1 second (i.e. $T_{\text{approx}} \approx 2\pi\sqrt{\ell/g} = 1$sec). Use your code from part (a) to calculate the actual period for your grandfather clock. How much time will you gain or lose per day?

Note: Figure credit, "Simple pendulum height" by Krishnavedala - Own work. Licensed under CC0 via Wikimedia Commons link.

APPROXIMATE INTEGRATION RULES

Left-endpoint rule: For $\Delta x = (b-a)/N$, and $x_i = a + i\Delta x$,

$$\int_a^b f(x)\,dx \approx L_N = \sum_{i=1}^{N} f(x_{i-1})\Delta x$$

Right-endpoint rule: For $\Delta x = (b-a)/N$, and $x_i = a + i\Delta x$,

$$\int_a^b f(x)\,dx \approx R_N = \sum_{i=1}^{N} f(x_i)\Delta x$$

Midpoint rule: For $\Delta x = (b-a)/N$, $x_i = a + i\Delta x$, and $\bar{x}_i = (x_{i-1} + x_i)/2$,

$$\int_a^b f(x)\,dx \approx M_N = \sum_{i=1}^{N} f(\bar{x}_i)\Delta x$$

Trapezoidal rule: For $\Delta x = (b-a)/N$, and $x_i = a + i\Delta x$,

$$\begin{aligned}\int_a^b f(x)\,dx \approx T_N &= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N)\right)\\ &= \frac{\Delta x}{2}\left(f(x_0) + f(x_N) + 2\sum_{i=1}^{N-1} f(x_i)\right)\end{aligned}$$

Simpson's rule: For $\Delta x = (b-a)/N$, and $x_i = a + i\Delta x$,

$$\begin{aligned}\int_a^b f(x)\,dx \approx S_N &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right)\\ &= \frac{\Delta x}{3}\left(f(x_0) + f(x_N) + 4\sum_{i=1}^{N/2} f(x_{2i-1}) + 2\sum_{i=1}^{(N-2)/2} f(x_{2i})\right)\end{aligned}$$