Math 251H Midterm 1: Topic Overview

Textbook sections

1.1-3, 2.1-2.6, 2.8, 3.1-3.8, 4.1-4.2

General theory

We started the class by discussing:

- What is a differential equation? What is an initial value problem (IVP)?
- Order of differential equations; linear vs nonlinear differential equations

First order differential equations

Theory:

For $L[y] = y' + p(t)y = g(t)$,

- **1st** order equation: has **1** linearly independent solution, IVP requires **1** initial condition.
- Solution domains; existence & uniqueness.

Solution-free analysis of first order equations: gain insight without solving! Sketch & interpret

- **Direction fields**
- Phase line for **autonomous equations**

Solution methods:

- Separable equations (linear or nonlinear equations that are separable).
- Integrating factors (linear equations only).
- Exact equations. **Note:** if an equation is not exact, we can sometimes still find an *integrating factor* to make it exact.

Applications:

Modeling including

- Mixing problems
- Population dynamics

You should be able to interpret word problems as linear first order ODEs.

Second order differential equations

Theory:

For $L[y] = y'' + p(t)y' + q(t)y = g(t)$,

- **2nd** order equation: has **2** linearly independent solutions, IVP requires **2** initial conditions.
- Linear independence and the Wronskian
- Solution domains; existence & uniqueness.

Solution methods:

For constant coefficient, $L[y] = ay'' + by' + cy = g(t)$

- Homogeneous equation $L[y] = 0$ ($g(t) = 0$): $y = e^{rt}$, find roots r.
  - $r_{1,2}$ real & distinct: general solution $y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
  - $r_1 = r_2 = r$, repeated root: general solution $y_h = C_1 e^{rt} + C_2 t e^{rt}$
  - $r_{1,2}$ complex, $r_{1,2} = \alpha \pm i\beta$: general solution $y_h = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$
- Inhomogeneous equations $L[y] = g(t)$ ($g(t) \neq 0$): Require
  1. *Homogeneous solution* $y_h(t)$, satisfies $L[y_h] = 0$, as above
  2. *Particular solution* $y_p(t)$ to satisfy inhomogeneity, $L[y_p] = g$
  3. Then the **general solution** is $y(t) = y_h(t) + y_p(t)$

  Get **particular solution** $y_p(t)$ using

  - Method of undetermined coefficients
  - Variation of parameters

Note: given a single solution $y_1(t)$ of the homogeneous equation $L[y] = 0$, you can use **reduction of order** to get a second, linearly independent solution.
For variable coefficient equations, $L[y] = y'' + p(t)y' + q(t)y = g(t)$, you can still:

- Use **reduction of order** to get a second, linearly independent solution, if you are given a single solution $y_1(t)$ of the homogeneous equation.
- Use **variation of parameters** to find a particular solution, if you are given the two, linearly independent solutions to the homogeneous equation, y_1 and y_2.

Applications:

Mechanical and electrical vibrations, including:

- Spring-mass systems $mx'' + bx' + cx = g(t)$, where $x(t)$ is displacement from equilibrium
- LCR circuits $LQ'' + RQ' + Q/C = E(t)$, where $Q(t)$ is the charge.

You should be able to interpret word problems as linear second order ODEs, interpret results in the context of these physical systems, sketch solutions, and interpret/identify sketches of solution.
In the context of these physical systems, we talked about:

- Underdamped, critically damped, and overdamped systems (homogeneous part)
- Natural vs quasi-frequency (homogeneous part)
- Resonance (when $g(t)$, $E(t) = A\cos(\beta t)$): forcing frequency β matches natural frequency ω.
- Beats (when $g(t)$, $E(t) = A\cos(\beta t)$)

Higher order differential equations

Theory:

For $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \ldots + p_{n-1}(t)y' + p_n(t)y = g(t)$,

- **n**$^{\text{th}}$order ODE: homogeneous eq. has **n** linearly independent solutions, IVP requires **n** initial conditions.
- Linear independence and the Wronskian
- Solution domains; existence & uniqueness.

Solution methods:

For constant coefficient, $L[y] = y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \ldots + a_{n-1}y' + a_ny = 0$

- Homogeneous equation $L[y] = 0$ ($g(t) = 0$): $y = e^{rt}$, find roots r. *See second order solution methods.*
- Inhomogeneous equations $L[y] = g(t)$ ($g(t) \neq 0$): *Outside scope of exam.*

For variable coefficient equations, $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_ny = g(t)$:
Outside scope of exam.

Applications:

Did not discuss applications.