B Aquaculture

Aquaculture is the art of cultivating the plants and animals indigenous to water. In the example considered here, it is assumed that a batch of catfish are raised in a pond. We are interested in determining the best time for harvesting the fish so that the cost per pound for raising the fish is minimized.

A differential equation describing the growth of fish may be expressed as

$$\text{(1)} \qquad \frac{dW}{dt} = KW^{\alpha} \ ,$$

where $W(t)$ is the weight of the fish at time t and K and α are empirically determined growth constants. The functional form of this relationship is similar to that of the growth models for other species. Modeling the growth rate or metabolic rate by a term like W^{α} is a common assumption. Biologists often refer to equation (1) as the **allometric equation.** It can be supported by plausibility arguments such as growth rate depending on the surface area of the gut (which varies like $W^{2/3}$) or depending on the volume of the animal (which varies like W).

(a) Solve equation (1) when $\alpha \neq 1$.

(b) The solution obtained in part (a) grows large without bound, but in practice there is some limiting maximum weight $W_{\max}$ for the fish. This limiting weight may be included in the differential equation describing growth by inserting a dimensionless variable S that can range between 0 and 1 and involves an empirically determined parameter μ. Namely, we now assume that

$$\text{(2)} \qquad \frac{dW}{dt} = KW^{\alpha}S \ ,$$

where $S \coloneqq 1 - (W/W_{\max})^{\mu}$. When $\mu = 1 - \alpha$, equation (2) has a closed form solution. Solve equation (2) when $K = 10$, $\alpha = 3/4$, $\mu = 1/4$, $W_{\max} = 81$ (ounces), and $W(0) = 1$ (ounce). The constants are given for t measured in months.

(c) The differential equation describing the total cost in dollars $C(t)$ of raising a fish for t months has one constant term K_1 that specifies the cost per month (due to costs such as interest, depreciation, and labor) and a second constant K_2 that multiplies the growth rate (because the amount of food consumed by the fish is approximately proportional to the growth rate). That is,

$$\text{(3)} \qquad \frac{dC}{dt} = K_1 + K_2\frac{dW}{dt} \ .$$

Solve equation (3) when $K_1 = 0.4$, $K_2 = 0.1$, $C(0) = 1.1$ (dollars), and $W(t)$ is as determined in part (b).

(d) Sketch the curve obtained in part (b) that represents the weight of the fish as a function of time. Next, sketch the curve obtained in part (c) that represents the total cost of raising the fish as a function of time.

(e) To determine the optimal time for harvesting the fish, sketch the ratio $C(t)/W(t)$. This ratio represents the total cost per ounce as a function of time. When this ratio reaches its minimum—that is, when the total cost per ounce is at its lowest—it is the optimal time to harvest the fish. Determine this optimal time to the nearest month.