C Transverse Vibrations of a Beam

In applying elasticity theory to study the transverse vibrations of a beam, one encounters the equation

$$EIy^{(4)}(x) - \gamma\lambda y(x) = 0 \ ,$$

where $y(x)$ is related to the displacement of the beam at position x; the constant E is Young's modulus; I is the area moment of inertia, which we assume is constant; γ is the constant mass per unit length of the beam; and λ is a positive parameter to be determined. We can simplify the equation by letting $r^4 := \gamma\lambda/EI$; that is, we consider

$$\textbf{(5)} \qquad \boldsymbol{y^{(4)}(x) - r^4 y(x) = 0} \ .$$

When the beam is clamped at each end, we seek a solution to (5) that satisfies the boundary conditions

$$\textbf{(6)} \qquad \boldsymbol{y(0) = y'(0) = 0 \quad \text{and} \quad y(L) = y'(L) = 0} \ ,$$

where L is the length of the beam. The problem is to determine those nonnegative values of r for which equation (5) has a nontrivial solution $(y(x) \not\equiv 0)$ that satisfies (6). To do this, proceed as follows:

(a) Show that there are no nontrivial solutions to the boundary value problem (5)–(6) when $r = 0$.

(b) Represent the general solution to (5) for $r > 0$ in terms of sines, cosines, hyperbolic sines, and hyperbolic cosines.

(c) Substitute the general solution obtained in part (b) into the equations (6) to obtain four linear algebraic equations for the four coefficients appearing in the general solution.

(d) Show that the system of equations in part (c) has nontrivial solutions only for those values of r satisfying

$$\textbf{(7)} \quad \cosh(rL) = \sec(rL) \ .$$

(e) On the same coordinate system, sketch the graphs of $\cosh(rL)$ and $\sec(rL)$ versus r for $L = 1$ and argue that equation (7) has an infinite number of positive solutions.

(f) For $L = 1$, determine the first two positive solutions to (7) numerically, and plot the corresponding solutions to the boundary value problem (5)–(6). [*Hint:* You may want to use Newton's method in Appendix B.]

D Higher-Order Difference Equations

Difference equations occur in mathematical models of physical processes and as tools in numerical analysis. The theory of linear difference equations parallels the theory of linear differential equations. Using the results of this chapter as models, both for the statements of theorems and for their proofs, we can develop a theory for linear difference equations.

A Kth order linear difference equation is an equation of the form

$$\textbf{(8)} \qquad a_K(n)y_{n+K} + a_{K-1}(n)y_{n+K-1} + \cdots + a_1(n)y_{n+1} + a_0(n)y_n = g_n, \quad n \geq 0,$$

where $a_K(n), \ldots, a_0(n)$, and g_n are defined for all nonnegative integers n. By a solution to (8) we mean a sequence of real numbers $\{y_n\}_{n=0}^{\infty}$ that satisfies (8) for all integers $n \geq 0$.