(d) In equations (5) and (6), make the change of variables $w = t - v$ and show that

$$y(t) = \int_0^t A'(w)g(t-w)\,dw \ , \tag{7}$$

$$y(t) = \int_0^t A(t-w)g'(w)\,dw + A(t)g(0) \ . \tag{8}$$

Equations (5)–(8) are referred to as **Duhamel's formulas,** in honor of the French mathematician J. M. C. Duhamel. These formulas are helpful in determining the response of the system to a general input $g(t)$, since the indicial admittance of the system can be determined experimentally by measuring the response of the system to a unit step function.

(e) The impulse response function $h(t)$ is defined as $h(t) \coloneqq \mathscr{L}^{-1}\{H\}(t)$, where $H(s)$ is the transfer function. Show that $h(t) = A'(t)$, so that equations (5) and (7) can be written in the form

$$y(t) = \int_0^t h(t-v)g(v)\,dv = \int_0^t h(v)g(t-v)\,dv \ . \tag{9}$$

We remark that the indicial admittance is the response of the system to a unit step function, and the impulse response function is the response to the unit impulse or delta function (see Section 7.8). But the delta function is the derivative (in a generalized sense) of the unit step function. Therefore, the fact that $h(t) = A'(t)$ is not really surprising.

B Frequency Response Modeling

Frequency response modeling of a linear system is based on the premise that the dynamics of a linear system can be recovered from a knowledge of how the system responds to sinusoidal inputs. (This will be made mathematically precise in Theorem 13.) In other words, to determine (or identify) a linear system, all one has to do is observe how the system reacts to sinusoidal inputs.

Let's assume that we have a linear system governed by

$$y'' + py' + qy = g(t) \ , \tag{10}$$

where p and q are real constants. The function $g(t)$ is called the **forcing function** or **input function.** When $g(t)$ is a sinusoid, the particular solution to (10) obtained by the method of undetermined coefficients is the **steady-state solution** or **output function** $y_{ss}(t)$ corresponding to $g(t)$. We can think of a linear system as a compartment or block into which goes an input function g and out of which comes the output function y_{ss} (see Figure 7.32). To **identify** a linear system means to determine the coefficients p and q in equation (10).

It will be convenient for us to work with complex variables. A complex number z is usually expressed in the form $z = \alpha + i\beta$, with α, β real numbers and i denoting $\sqrt{-1}$. We can also express z in polar form, $z = re^{i\theta}$, where $r^2 = \alpha^2 + \beta^2$ and $\tan\theta = \beta/\alpha$. Here $r\ (\geq 0)$ is called the **magnitude** and θ the **phase angle** of z.

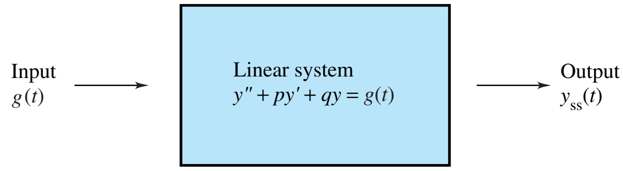

Figure 7.32 Block diagram depicting a linear system

The following theorem gives the relationship between the linear system and its response to sinusoidal inputs in terms of the **transfer function** $H(s)$ [see Project A, equation (2)].

Steady-State Solutions to Sinusoidal Inputs

Theorem 13. Let $H(s)$ be the transfer function for equation (10). If $H(s)$ is finite at $s = i\omega$, with ω real, then the steady-state solution to (10) for $g(t) = e^{i\omega t}$ is

$$\textbf{(11)} \qquad y_{ss}(t) = H(i\omega)e^{i\omega t} = H(i\omega)\{\cos \omega t + i \sin \omega t\} .$$

(a) Prove Theorem 13. $\left[\textit{Hint:}\text{ Guess } y_{ss}(t) = Ae^{i\omega t} \text{ and show that } A = H(i\omega).\right]$

(b) Use Theorem 13 to show that if $g(t) = \sin \omega t$, then the steady-state solution to (10) is $M(\omega)\sin\left[\omega t + N(\omega)\right]$, where $H(i\omega) = M(\omega)e^{iN(\omega)}$ is the polar form for $H(i\omega)$.

(c) Solve for $M(\omega)$ and $N(\omega)$ in terms of p and q.

(d) Experimental results for modeling done by frequency response methods are usually presented in **frequency response**[†] or **Bode plots.** There are two types of Bode plots. The first is the log of the magnitude $M(\omega)$ of $H(i\omega)$ versus the angular frequency ω using a log scale for ω. The second is a plot of the phase angle or argument $N(\omega)$ of $H(i\omega)$ versus the angular frequency using a log scale for ω. The Bode plots for the transfer function $H(s) = (1 + 0.2s + s^2)^{-1}$ are given in Figure 7.33.

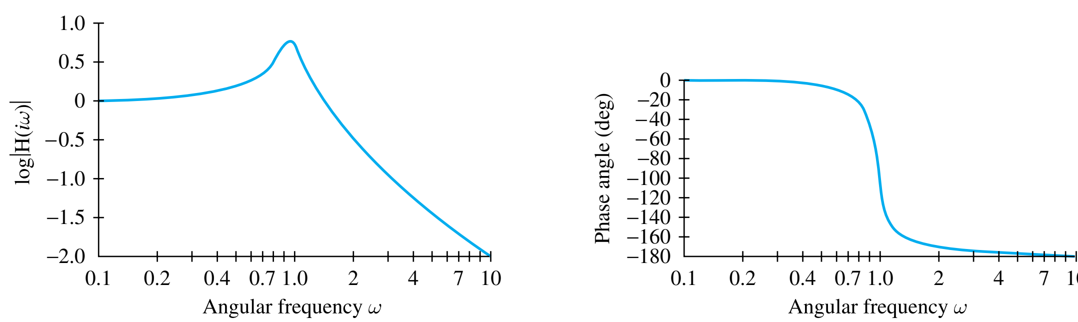

Figure 7.33 Bode plots for $H(i\omega) = \left[1 + 0.2(i\omega) + (i\omega)^2\right]^{-1}$

Sketch the Bode plots of the linear system governed by equation (10) with $p = 0.4$ and $q = 1.0$. Use $\omega = 0.3, 0.6, 0.9, 1.2$, and 1.5 for the plot of $M(\omega)$ and $\omega = 0.5, 0.8, 1, 2$, and 5 for the plot of $N(\omega)$.

(e) Assume we know that $q = 1$. When we input a sine wave with $\omega = 2$, the system settles into a steady-state sinusoidal output with magnitude $M(2) = 0.333$. Find p and thus identify the linear system.

(f) Suppose a sine wave input with $\omega = 2$ produces a steady-state sinusoidal output with magnitude $M(2) = 0.5$ and that when $\omega = 4$, then $M(4) = 0.1$. Find p and q and thus identify the system.

[†]Frequency response curves are also discussed in Section 4.10.

In most applications there are some inaccuracies in the measurement of the magnitudes and frequencies. To compensate for these errors, sinusoids with several different frequencies are used as input. A least-squares approximation for p and q is then found. For a discussion of frequency response modeling as a mathematical modeling tool, see the chapter by W. F. Powers, "Modeling Linear Systems by Frequency Response Methods," in *Differential Equations Models,* by M. Braun, C. Coleman, and D. Drew (eds.) (Springer-Verlag, New York, 1983), Chapter 9. Additional examples may be found in *Schaum's Outline on Feedback and Control Systems,* by J. J. DiStefano, A. R. Stubberud, and I. J. Williams (McGraw-Hill, New York, 1995, 3rd edition), Chapter 15.

C Determining System Parameters

In mechanical design one sometimes must determine system parameters even though information on the system forces is incomplete. For example, the differential equation governing the motion $x(t)$ of an externally forced damped mass–spring oscillator was shown in Section 4.1 (page 153) to be

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t) \ ,$$

where $f(t)$ is the external force; the other parameters are defined in Section 4.1. Assume the system is underdamped ($b^2 < 4mk$; see Section 4.9) and starts from rest $\left[x(0) = 0, x'(0) = 0\right]$ and that the force is bounded: $|f(t)| \le A$ for all t.

(a) Show that the transforms $X(s)$ and $F(s)$ of $x(t)$ and $f(t)$ are related by

$$X(s) = \frac{1}{ms^2 + bs + k}F(s) \ .$$

(b) Use the convolution theorem to derive the formula

$$x(t) = \frac{1}{\beta m}\int_0^t f(t - v)e^{-bv/2m}\sin \beta v \, dv \ ,$$

where

$$\beta := \frac{1}{2m}\sqrt{4mk - b^2} \ .$$

(c) Show that the motion $x(t)$ is bounded under these circumstances by

$$|x(t)| \le At/\beta m$$

and also by

$$|x(t)| \le 2A/\beta b \ .$$

(d) Suppose the mass $m = 5$ kg, the spring constant $k = 3000$ N/m, and the force is bounded by $A = 10$ N. What range of values for the damping constant b will ensure a displacement of 1 cm (0.01 m) or less?