(f) The force on a mass–spring system suspended vertically in a gravitational field was shown in Section 4.10 (page 228) to be $F = -ky + mg$. Derive the Hamiltonian and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

(g) As indicated in Section 4.8 (page 209), the Duffing spring force is modeled by $F = -y - y^3$. Derive the Hamiltonian and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

(h) For the pendulum system studied in Section 4.8, Example 8, the force is given by (cf. Figure 4.18, page 210)

$$F = -\ell mg \sin\theta = -\frac{\partial}{\partial\theta}(-\ell mg\cos\theta) = -\frac{\partial}{\partial\theta}V(\theta)$$

(where ℓ is the length of the pendulum). For angular variables, the Hamiltonian formulation dictates expressing the *angular velocity variable* θ' *in terms of the angular momentum* $p = m\ell^2\theta'$; the kinetic energy, mass $\times$ velocity$^2/2$, is expressed as $m(\ell\theta')^2/2 = p^2/(2m\ell^2)$. Derive the Hamiltonian for the pendulum and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

(i) The Coulomb force field is a force that varies as the reciprocal square of the distance from the origin: $F = k/y^2$. The force is *attractive* if $k < 0$ and *repulsive* if $k > 0$. Sketch the phase plane trajectories for this motion. Sketch the trajectories when damping is present.

(j) For an attractive Coulomb force field, what is the *escape velocity* for a particle situated at a position y? That is, what is the minimal (outward-directed) velocity required for the trajectory to reach $y = \infty$?

E Cleaning Up the Great Lakes

A simple mathematical model that can be used to determine the time it would take to clean up the Great Lakes can be developed using a multiple compartmental analysis approach.† In particular, we can view each lake as a tank that contains a liquid in which is dissolved a particular pollutant (DDT, phosphorus, mercury). Schematically, we view the lakes as consisting of five tanks connected as indicated in Figure 5.55.

Figure 5.55 Compartmental model of the Great Lakes with flow rates (mi^3/yr) and volumes (mi^3)

†For a detailed discussion of this model, see *An Introduction to Mathematical Modeling* by Edward A. Bender (Krieger, New York, 1991), Chapter 8.

For our model, we make the following assumptions:

1. The volume of each lake remains constant.
2. The flow rates are constant throughout the year.
3. When a liquid enters the lake, perfect mixing occurs and the pollutants are uniformly distributed.
4. Pollutants are dissolved in the water and enter or leave by inflow or outflow of solution.

Before using this model to obtain estimates on the cleanup times for the lakes, we consider some simpler models:

(a) Use the outflow rates given in Figure 5.55 to determine the time it would take to "drain" each lake. This gives a lower bound on how long it would take to remove all the pollutants.
(b) A better estimate is obtained by assuming that each lake is a separate tank with *only* clean water flowing in. Use this approach to determine how long it would take the pollution level in each lake to be reduced to 50% of its original level. How long would it take to reduce the pollution to 5% of its original level?
(c) Finally, to take into account the fact that pollution from one lake flows into the next lake in the chain, use the entire multiple compartment model given in Figure 5.55 to determine when the pollution level in each lake has been reduced to 50% of its original level, assuming pollution has ceased (that is, inflows not from a lake are clean water). Assume that all the lakes initially have the same pollution concentration p. How long would it take for the pollution to be reduced to 5% of its original level?

F A Growth Model for Phytoplankton—Part I

Courtesy of Dr. Olivier Bernard and Dr. Jean-Luc Gouzé, INRIA

A chemostat is a stirred tank in which phytoplankton grow by consuming a nutrient (e.g., nitrate). The nutrient is supplied to the tank at a given rate, and a solution containing the phytoplankton and remaining nutrient is removed at an equal rate (cf. Figure 5.56). The chemostat reproduces *in vitro* the conditions of the growth of phytoplankton in the ocean; the phytoplankton is the first element of the marine food chain.

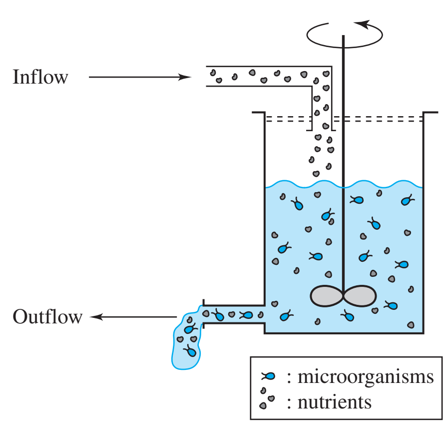

Figure 5.56 Chemostat