opposite the gravitational force. Introduce a new variable, $\zeta$, that gives the displacement of the object from its equilibrium position $l$ (that is, $z = \zeta + l$). You can now write the ODE in a more familiar form. [*Hint:* Recall the mass–spring system and the equilibrium case.] Now you should recognize the type of solution for this problem. What is the natural frequency?

**(c)** In this task you consider the effect of friction. The bobbing object is a cube, 1 ft on a side, that weighs 32 lb. Let $\gamma_w = 3$ lb-sec/ft, $\rho = 62.57$ lb/ft$^3$, and suppose the object is initially placed on the surface of the water. Solve the governing ODE by hand to find the general solution. Next, find the particular solution for the case in which the cube is initially placed on the surface of the water and is given no initial velocity. Provide a plot of the position of the object as a function of time $t$.

**(d)** In this step of the project, you develop a numerical solution to the same problem presented in part (c). The numerical solution will be useful (indeed necessary) for subsequent parts of the project. This case provides a trial to verify that your numerical solution is correct. Go back to the initial ODE you developed in part (a). Using parameter values given in part (c), solve the initial value problem for the cube starting on the surface with no initial velocity. To solve this problem numerically, you will have to write the second-order ODE as a system of two first-order ODEs, one for vertical position $z$ and one for vertical velocity $w$. Plot your results for vertical position as a function of time $t$ for the first 3 or 4 sec and compare with the analytical solution you found in part (c). Are they in close agreement? What might you have to do in order to compare these solutions? Provide a plot of both your analytical and numerical solutions on the same graph.

**(e)** Suppose a sphere of radius $R$ is allowed to bob in the water. Derive the governing second-order equation for the sphere using Archimedes' principle and allowing for friction due to its motion in the water. Suppose a sphere weighs 32 lb, has a radius of $1/2$ ft, and $\gamma_w = 3.0$ lb-sec/ft. Determine the limiting value of the position of the sphere without solving the ODE. Next, solve the governing ODE for the velocity and position of the sphere as a function of time for a sphere placed on the surface of the water. You will need to write the governing second-order ODE as a system of two ODEs, one for velocity and one for position. What is the limiting position of the sphere for your solution? Does it agree with the equilibrium solution you found above? How does it compare with the equilibrium position of the cube? If it is different, explain why.

**(f)** Suppose the sphere in part (d) is a volleyball. Calculate the position of the sphere as a function of time $t$ for the first 3 sec if the ball is submerged so that its lowest point is 5 ft under water. Will the ball leave the water? How high will it go? Next, calculate the ball's trajectory for initial depths lower than 5 ft and higher than 5 ft. Provide plots of velocity and position for each case and comment on what you see. Specifically, comment on the relationship between the initial depth of the ball and the maximum height the ball eventually attains.

You might consider taking a volleyball into a swimming pool to gather real data in order to verify and improve on your model. If you do so, report the data you found and explain how you used it for verification and improvement of your model.

## D Hamiltonian Systems

The problems in this project explore the **Hamiltonian**[†] **formulation** of the laws of motion of a system and its phase plane implications. This formulation replaces Newton's second law $F = ma = my''$ and is based on three mathematical manipulations:

[†] *Historical Footnote:* Sir William Rowan Hamilton (1805–1865) was an Irish mathematical physicist. Besides his work in mechanics, he invented quaternions and discovered the anticommutative law for vector products.

(i) It is presumed that the force $F(t, y, y')$ depends only on $y$ and has an antiderivative $-V(y)$, that is, $F = F(y) = -dV(y)/dy$.
(ii) The velocity variable $y'$ is replaced throughout by the momentum $p = my'$ (so $y' = p/m$).
(iii) The **Hamiltonian** of the system is defined as

$$H = H(y, p) = \frac{p^2}{2m} + V(y) \ .$$

**(a)** Express Newton's law $F = my''$ as an equivalent first-order system in the manner prescribed in Section 5.3.
**(b)** Show that this system is equivalent to **Hamilton's equations**

$$\frac{dy}{dt} = \frac{\partial H}{\partial p} \qquad \left(= \frac{p}{m}\right) , \tag{2}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial y} \qquad \left(= -\frac{dV}{dy}\right) . \tag{3}$$

**(c)** Using Hamilton's equations and the chain rule, show that the Hamiltonian remains constant along the solution curves:

$$\frac{d}{dt}H(y, p) = 0 \ .$$

In the formula for the Hamiltonian function $H(y, p)$, the first term, $p^2/(2m) = m(y')^2/2$, is the **kinetic energy** of the mass. By analogy, then, the second term $V(y)$ is known as the **potential energy** of the mass, and the Hamiltonian is the total energy. The total (mechanical)† energy is constant—hence "conserved"—when the forces $F(y)$ do not depend on time $t$ or velocity $y'$; such forces are called **conservative.** The energy integral lemma of Section 4.8 (page 204) is simply an alternate statement of the conservation of energy.

Hamilton's formulation for mechanical systems and the conservation of energy principle imply that the phase plane trajectories of conservative systems lie on the curves where the Hamiltonian $H(y, p)$ is constant, and plotting these curves may be considerably easier than solving for the trajectories directly (which, in turn, is easier than solving the original system!).

**(d)** For the mass–spring oscillator of Section 4.1, the spring force is given by $F = -ky$ (where $k$ is the spring constant). Find the Hamiltonian, express Hamilton's equations, and show that the phase plane trajectories $H(y, p) =$ constant for this system are the ellipses given by $p^2/(2m) + ky^2/2 =$ constant. See Figure 5.14, page 270.

The **damping** force $-by'$ considered in Section 4.1 is not conservative, of course. Physically speaking, we know that damping drains the energy from a system until it grinds to a halt at an equilibrium point. In the phase plane, we can qualitatively describe the trajectory as continuously migrating to successively lower constant-energy orbits; stable centers become asymptotically stable spiral points when damping is taken into consideration.

**(e)** The second Hamiltonian equation (3), which effectively states $p' = my'' = F$, has to be changed to

$$p' = -\frac{\partial H}{\partial y} - by' = -\frac{\partial H}{\partial y} - \frac{bp}{m}$$

when damping is present. Show that the Hamiltonian decreases along trajectories in this case (for $b > 0$):

$$\frac{d}{dt}H(y, p) = -b\left(\frac{p}{m}\right)^2 = -b(y')^2 \ .$$

†Physics states that when all forms of energy, such as heat and radiation, are taken into account, energy is conserved even when the forces are not conservative.

**(f)** The force on a mass–spring system suspended vertically in a gravitational field was shown in Section 4.10 (page 228) to be $F = -ky + mg$. Derive the Hamiltonian and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

**(g)** As indicated in Section 4.8 (page 209), the Duffing spring force is modeled by $F = -y - y^3$. Derive the Hamiltonian and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

**(h)** For the pendulum system studied in Section 4.8, Example 8, the force is given by (cf. Figure 4.18, page 210)

$$F = -\ell mg \sin\theta = -\frac{\partial}{\partial\theta}(-\ell mg\cos\theta) = -\frac{\partial}{\partial\theta}V(\theta)$$

(where $\ell$ is the length of the pendulum). For angular variables, the Hamiltonian formulation dictates expressing the *angular velocity variable* $\theta'$ *in terms of the angular momentum* $p = m\ell^2\theta'$; the kinetic energy, mass $\times$ velocity$^2/2$, is expressed as $m(\ell\theta')^2/2 = p^2/(2m\ell^2)$. Derive the Hamiltonian for the pendulum and sketch the phase plane trajectories. Sketch the trajectories when damping is present.

**(i)** The Coulomb force field is a force that varies as the reciprocal square of the distance from the origin: $F = k/y^2$. The force is *attractive* if $k < 0$ and *repulsive* if $k > 0$. Sketch the phase plane trajectories for this motion. Sketch the trajectories when damping is present.

**(j)** For an attractive Coulomb force field, what is the *escape velocity* for a particle situated at a position $y$? That is, what is the minimal (outward-directed) velocity required for the trajectory to reach $y = \infty$?

## E Cleaning Up the Great Lakes

A simple mathematical model that can be used to determine the time it would take to clean up the Great Lakes can be developed using a multiple compartmental analysis approach.[†] In particular, we can view each lake as a tank that contains a liquid in which is dissolved a particular pollutant (DDT, phosphorus, mercury). Schematically, we view the lakes as consisting of five tanks connected as indicated in Figure 5.55.

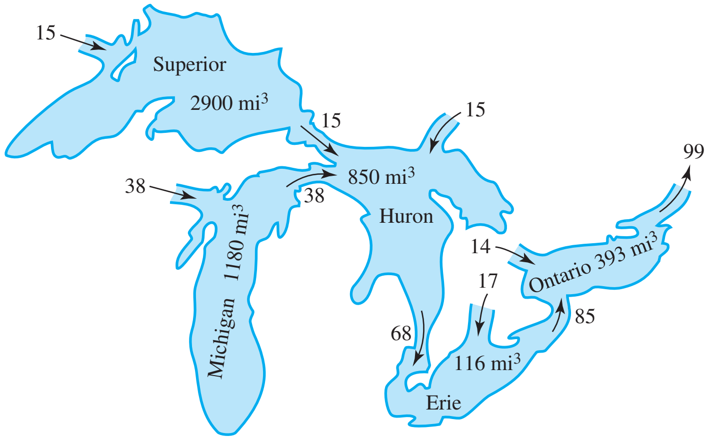


**Figure 5.55** Compartmental model of the Great Lakes with flow rates (mi$^3$/yr) and volumes (mi$^3$)

[†]For a detailed discussion of this model, see *An Introduction to Mathematical Modeling* by Edward A. Bender (Krieger, New York, 1991), Chapter 8.