G Market Equilibrium: Stability and Time Paths

Courtesy of James E. Foster, George Washington University

A *perfectly competitive market* is made up of many buyers and sellers of an economic product, each of whom has no control over the market price. In this model, the overall quantity demanded by the buyers of the product is taken to be a function of the price of the product (among other things) called the *demand function*. Similarly, the overall quantity supplied by the sellers of the product is a function of the price of the product (among other things) called the *supply function*. A market is in *equilibrium* at a price where the quantity demanded is just equal to the quantity supplied.

The *linear* model assumes that the demand and supply functions have the form $q_d = d_0 - d_1 p$ and $q_s = -s_0 + s_1 p$, respectively, where p is the market price of the product, q_d is the associated quantity demanded, q_s is the associated quantity supplied, and d_0, d_1, s_0, and s_1 are all positive constants. The functional forms ensure that the "laws" of downward sloping demand and upward sloping supply are being satisfied. It is easy to show that the equilibrium price is $p^* = (d_0 + s_0)/(d_1 + s_1)$.

Economists typically assume that markets are in equilibrium and justify this assumption with the help of stability arguments. For example, consider the simple *price adjustment equation*

$$\frac{dp}{dt} = \lambda(q_d - q_s) \ ,$$

where $\lambda > 0$ is a constant indicating the *speed of adjustments*. This follows the intuitive requirement that price rises when demand exceeds supply and falls when supply exceeds demand. The market equilibrium is said to be *globally stable* if, for every initial price level $p(0)$, the price adjustment path $p(t)$ satisfies $p(t) \to p^*$ as $t \to \infty$.

(a) Find the price adjustment path: Substitute the expressions for q_d and q_s into the price adjustment equation and show that the solution to the resulting differential equation is $p(t) = [p(0) - p^*]e^{ct} + p^*$, where $c = -\lambda(d_1 + s_1)$.

(b) Is the market equilibrium globally stable?

Now consider a model that takes into account the *expectations* of agents. Let the market demand and supply functions over time $t \geq 0$ be given by

$$q_d(t) = d_0 - d_1 p(t) + d_2 p'(t) \quad \text{and} \quad q_s(t) = -s_0 + s_1 p(t) - s_2 p'(t) \ ,$$

respectively, where $p(t)$ is the market price of the product, $q_d(t)$ is the associated quantity demanded, $q_s(t)$ is the associated quantity supplied, and d_0, d_1, d_2, s_0, s_1, and s_2 are all positive constants. The functional forms ensure that, when faced with an increasing price, demanders will tend to purchase more (before prices rise further) while suppliers will tend to offer less (to take advantage of the higher prices in the future). Now given the above stability argument, we restrict consideration to *market clearing time paths* $p(t)$ satisfying $q_d(t) = q_s(t)$, for all $t \geq 0$, and explore the evolution of price over time. We say that the market is in *dynamic equilibrium* if $p'(t) = 0$ for all t. It is easy to show that the dynamic equilibrium in this model is given by

$p(t) = p^*$ for all t, where p^* is the market equilibrium price defined above. However, many other market clearing time paths are possible.

(c) Find a market clearing time path: Equate $q_d(t)$ and $q_s(t)$ and solve the resulting differential equation $p(t)$ in terms of its initial value $p_0 = p(0)$.
(d) Is it true that for any market clearing time path we must have $p(t) \to p^*$ as $t \to \infty$?
(e) If the price $p(t)$ of a product is \$5 at $t = 0$ months and demand and supply functions are modeled as $q_d(t) = 30 - 2p(t) + 4p'(t)$ and $q_s(t) = -20 + p(t) - 6p'(t)$, what will be the price after 10 months? As t becomes very large? What is happening to $p'(t)$ and how are the expectations of demanders and suppliers evolving?

For further reading, see, for example, *Mathematical Economics*, 2nd ed. by Akira Takayama (Cambridge University Press, Cambridge, 1985), Chapter 3; and *Fundamental Methods of Mathematical Economics*, 4th ed. by Alpha Chiang, and Kevin Wainwright (McGraw-Hill/Irvin, Boston, 2008), Chapter 14.

H Stability of Numerical Methods

Numerical methods are often tested on simple initial value problems of the form

$$y' + \lambda y = 0 \ , \qquad y(0) = 1 \ , \qquad (\lambda = \text{constant}) \ , \tag{8}$$

which has the solution $\phi(x) = e^{-\lambda x}$. Notice that for each $\lambda > 0$ the solution $\phi(x)$ tends to zero as $x \to +\infty$. Thus, a desirable property for any numerical scheme that generates approximations $y_0, y_1, y_2, y_3, \ldots$ to $\phi(x)$ at the points $0, h, 2h, 3h, \ldots$ is that, for $\lambda > 0$,

$$y_n \to 0 \quad \text{as} \quad n \to \infty \ . \tag{9}$$

For single-step linear methods, property (9) is called **absolute stability.**

(a) Show that for $x_n = nh$, Euler's method, when applied to the initial value problem (8), yields the approximations

$$y_n = (1 - \lambda h)^n \ , \qquad n = 0, 1, 2, \ldots ,$$

and deduce that this method is absolutely stable only when $0 < \lambda h < 2$. (This means that for a given $\lambda > 0$, we must choose the step size h sufficiently small in order for property (9) to hold.) Further show that for $h > 2/\lambda$, the error $y_n - \phi(x_n)$ grows large exponentially!
(b) Show that for $x_n = nh$ the trapezoid scheme of Section 3.6, applied to problem (8), yields the approximations

$$y_n = \left(\frac{1 - \lambda h/2}{1 + \lambda h/2}\right)^n \ , \qquad n = 0, 1, 2, \ldots ,$$

and deduce that this scheme is absolutely stable for all $\lambda > 0$, $h > 0$.
(c) Show that the improved Euler's method applied to problem (8) is absolutely stable for $0 < \lambda h < 2$.

Multistep Methods. When multistep numerical methods are used, instability problems may arise that cannot be circumvented by simply choosing a sufficiently small step size h. This is because multistep methods yield "extraneous solutions," which may dominate the calculations. To see what can happen, consider the two-step method

$$y_{n+1} = y_{n-1} + 2hf(x_n, y_n) \ , \qquad n = 1, 2, \ldots , \tag{10}$$

for the equation $y' = f(x, y)$.