Group Projects for Chapter 2

A Oil Spill in a Canal

In 1973 an oil barge collided with a bridge in the Mississippi River, leaking oil into the water at a rate estimated at 50 gallons per minute. In 1989 the Exxon Valdez spilled an estimated 11,000,000 gallons of oil into Prudhoe Bay in 6 hours†*, and in 2010 the Deepwater Horizon well leaked into the Gulf of Mexico at a rate estimated to be 15,000 barrels per day*†† *(1 barrel = 42 gallons)*. In this project you are going to use differential equations to analyze a simplified model of the dissipation of heavy crude oil spilled at a rate of S ft^3/sec into a flowing body of water. The flow region is a canal, namely a straight channel of rectangular cross section, w feet wide by d feet deep, having a constant flow rate of v ft/sec; the oil is presumed to float in a thin layer of thickness s (feet) on top of the water, without mixing.

In Figure 2.12, the oil that passes through the cross-section window in a short time Δt occupies a box of dimensions s by w by $v\Delta t$. To make the analysis easier, presume that the canal is conceptually partitioned into cells of length L ft. each, and that *within each particular cell* the oil instantaneously disperses and forms a *uniform* layer of thickness $s_i(t)$ in cell i (cell 1 starts at the point of the spill). So, at time t, the ith cell contains $s_i(t)wL$ ft^3 of oil. Oil flows out of cell i at a rate equal to $s_i(t)wv$ ft^3/sec, and it flows into cell i at the rate $s_{i-1}(t)wv$; it flows into the first cell at S ft^3/sec.

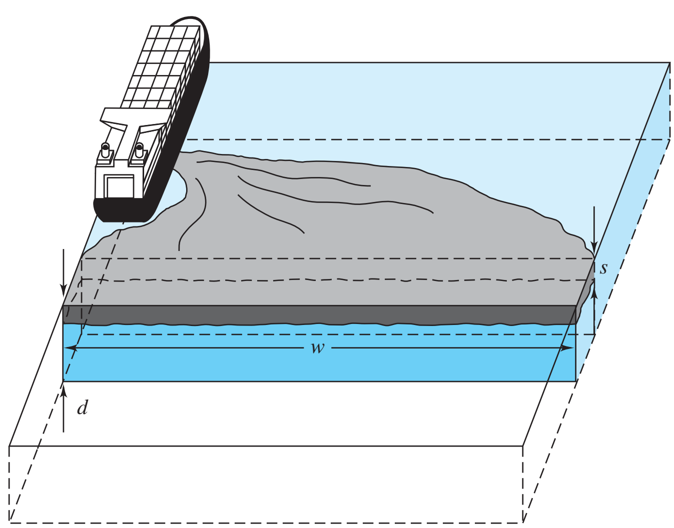

Figure 2.12 Oil leak in a canal.

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†Cutler J. Cleveland (Lead Author); C Michael Hogan and Peter Saundry (Topic Editor). 2010. "Deepwater Horizon oil spill." In: *Encyclopedia of Earth*, ed. Cutler J. Cleveland (Washington, D.C.: Environmental Information Coalition, National Council for Science and the Environment).

††Cutler J. Cleveland (Contributing Author); National Oceanic and Atmospheric Administration (Content source); Peter Saundry (Topic Editor). 2010. "Exxon Valdez oil spill." In: *Encyclopedia of Earth*, ed. Cutler J. Cleveland (Washington, D.C.: Environmental Information Coalition, National Council for Science and the Environment).

(a) Formulate a system of differential equations and initial conditions for the oil thickness in the first three cells. Take $S = 50$ gallons/min, which was roughly the spillage rate for the Mississippi River incident, and take $w = 200$ ft, $d = 25$ ft, and $v = 1$ mi/hr (which are reasonable estimates for the Mississippi River†). Take $L = 1000$ ft.
(b) Solve for $s_1(t)$. [*Caution*: Make sure your units are consistent.]
(c) If the spillage lasts for T seconds, what is the maximum oil layer thickness in cell 1?
(d) Solve for $s_2(t)$. What is the maximum oil layer thickness in cell 2?
(e) Probably the least tenable simplification in this analysis lies in regarding the layer thickness as uniform over distances of length L. Reevaluate your answer to part (**c**) with L reduced to 500 ft. By what fraction does the answer change?

B Differential Equations in Clinical Medicine

Courtesy of Philip Crooke, Vanderbilt University

In medicine, **mechanical ventilation** is a procedure that assists or replaces spontaneous breathing for critically ill patients, using a medical device called a *ventilator.* Some people attribute the first mechanical ventilation to Andreas Vesalius in 1555. Negative pressure ventilators (iron lungs) came into use in the 1940s–1950s in response to *poliomyelitis* (polio) epidemics. Philip Drinker and Louis Shaw are credited with its invention. Modern ventilators use positive pressure to inflate the lungs of the patient. In the ICU (intensive care unit), common indications for the initiation of mechanical ventilation are acute respiratory failure, acute exacerbation of chronic obstructive pulmonary disease, coma, and neuromuscular disorders. The goals of mechanical ventilation are to provide oxygen to the lungs and to remove carbon dioxide.

In this project, we model the mechanical process performed by the ventilator. We make the following assumptions about this process of filling the lungs with air and then letting them deflate to some rest volume (see Figure 2.13).

(i) The length (in seconds) of each breath is fixed (t_{tot}) and is set by the clinician, with each breath being identical to the previous breath.
(ii) Each breath is divided into two parts: *inspiration* (air flowing into the patient) and *expiration* (air flowing out of the patient). We assume that inspiration takes place over the interval $[0, t_i]$ and expiration over the time interval $[t_i, t_{tot}]$. The time t_i is called the *inspiratory time.*

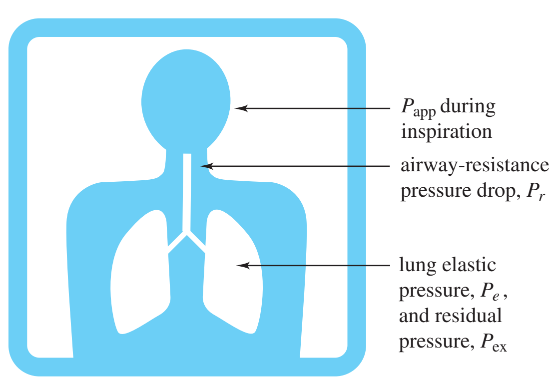

Figure 2.13 Lung ventilation pressures

†http://www.nps.gov/miss/riverfacts.htm