(d) Show that for the initial value problem

(11) $$y' + 2y = 0 \ , \qquad y(0) = 1 \ ,$$

the recurrence formula (10), with $x_n = nh$, becomes

(12) $$y_{n+1} + 4hy_n - y_{n-1} = 0 \ .$$

Equation (12), which is called a **difference equation,** can be solved by using the following approach. We postulate a solution of the form $y_n = r^n$, where r is a constant to be determined.

(e) Show that substituting $y_n = r^n$ in (12) leads to the "characteristic equation"

$$r^2 + 4hr - 1 = 0 \ ,$$

which has roots

$$r_1 = -2h + \sqrt{1 + 4h^2} \quad \text{and} \quad r_2 = -2h - \sqrt{1 + 4h^2} \ .$$

By analogy with the theory for second-order differential equations, it can be shown that a general solution of (12) is

$$y_n = c_1 r_1^n + c_2 r_2^n \ ,$$

where c_1 and c_2 are arbitrary constants. Thus, the general solution to the difference equation (12) has two independent constants, whereas the differential equation in (11) has only one, namely, $\phi(x) = ce^{-2x}$.

(f) Show that for each $h > 0$,

$$\lim_{n\to\infty} r_1^n = 0 \quad \text{but} \quad \lim_{n\to\infty} |r_2^n| = \infty \ .$$

Hence, the term r_1^n behaves like the solution $\phi(x_n) = e^{-2x_n}$ as $n \to \infty$. However, the extraneous solution r_2^n grows large without bound.

(g) Applying the scheme of (10) to the initial value problem (11) requires two starting values y_0, y_1. The exact values are $y_0 = 1$, $y_1 = e^{-2h}$. However, regardless of the choice of starting values and the size of h, the term $c_2 r_2^n$ will eventually dominate the full solution to the recurrence equation as x_n increases. Illustrate this instability taking $y_0 = 1$, $y_1 = e^{-2h}$, and using a calculator or computer to compute $y_2, y_3, \ldots, y_{100}$ from the recurrence formula (12) for $h = 0.5$ and $h = 0.05$. (*Note:* Even if initial conditions are chosen so that $c_2 = 0$, round-off error will inevitably "excite" the extraneous dominant solution.)

I Period Doubling and Chaos

In the study of dynamical systems, the phenomena of **period doubling** and **chaos** are observed. These phenomena can be seen when one uses a numerical scheme to approximate the solution to an initial value problem for a nonlinear differential equation such as the following logistic model for population growth:

(13) $$\frac{dp}{dt} = 10p(1 - p) \ , \qquad p(0) = 0.1 \ .$$

(See Section 3.2.)

(a) Solve the initial value problem (13) and show that $p(t)$ approaches 1 as $t \to +\infty$.

(b) Show that using Euler's method (see Sections 1.4 and 3.6) with step size h to approximate the solution to (13) gives

(14) $$p_{n+1} = (1 + 10h)p_n - (10h)p_n^2 \ , \qquad p_0 = 0.1 \ .$$

(c) For $h = 0.18, 0.23, 0.25$, and 0.3, show that the first 40 iterations of (14) appear to (i) converge to 1 when $h = 0.18$, (ii) jump between 1.18 and 0.69 when $h = 0.23$, (iii) jump between 1.23, 0.54, 1.16, and 0.70 when $h = 0.25$, and (iv) display no discernible pattern when $h = 0.3$.

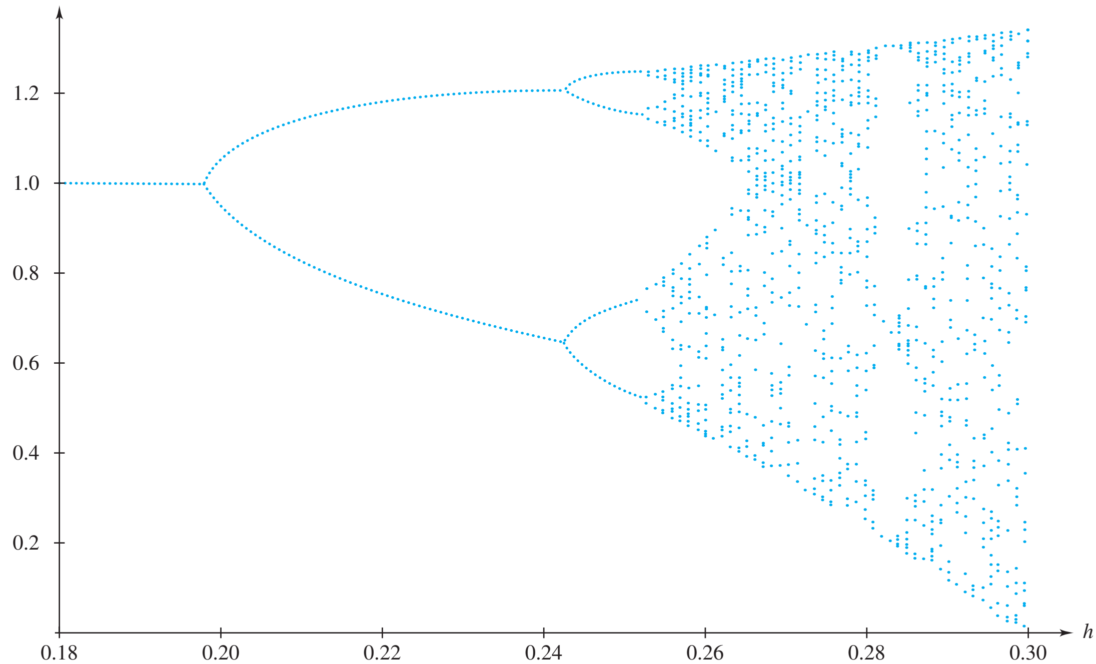

Figure 3.22 Period doubling to chaos

The transitions from convergence to jumping between two numbers, then four numbers, and so on, are called **period doubling.** The phenomenon displayed when $h = 0.3$ is referred to as **chaos.** This transition from period doubling to chaos as h increases is frequently observed in dynamical systems.

The transition to chaos is nicely illustrated in the bifurcation diagram (see Figure 3.22). This diagram is generated for equation (14) as follows. Beginning at $h = 0.18$, compute the sequence $\{p_n\}$ using (14) and, starting at $n = 201$, plot the next 30 values—that is, $p_{201}, p_{202}, \ldots, p_{230}$. Next, increment h by 0.001 to 0.181 and repeat. Continue this process until $h = 0.30$. Notice how the figure splits from one branch to two, then four, and then finally gives way to chaos.

Our concern is with the instabilities of the numerical procedure when h is not chosen small enough. Fortunately, the instability observed for Euler's method—the period doubling and chaos—was immediately recognized because we know that this type of behavior is not expected of a solution to the logistic equation. Consequently, if we had tried Euler's method with $h = 0.23$, 0.25, or 0.3 to solve (13) numerically, we would have realized that h was not chosen small enough.

The situation for the classical fourth-order Runge–Kutta method (see Section 3.7) is more troublesome. It may happen that for a certain choice of h period doubling occurs, but it is also possible that for other choices of h the numerical solution actually converges to a limiting value that is *not* the limiting value for any solution to the logistic equation in (13).

(d) Approximate the solution to (13) by computing the first 60 iterations of the classical fourth-order Runge–Kutta method using the step size $h = 0.3$. (Thus, for the subroutine on page 144, input $N = 60$ and $c = 60(0.3) = 18$.) Repeat with $h = 0.325$ and $h = 0.35$. Which values of h (if any) do you feel are giving the "correct" approximation to the solution? Why?

A further discussion of chaos appears in Section 5.8.