Several authors have devised improvements to Quicksort to reduce the probability of worst-case scenarios. (For example, instead of choosing one word at random, one chooses three words and uses the median for comparison.) The article "An Asymptotic Theory for Cauchy-Euler Differential Equations with Applications to the Analysis of Algorithms," by H.-H. Chern, H.-K. Hwang, and T.-H. Tsai (*Journal of Algorithms,* 44 (2002); 177–225) shows how the methodology in this project can be extended to analyze many of these strategies.

B Spherically Symmetric Solutions to Schrödinger's Equation for the Hydrogen Atom

In quantum mechanics one is interested in determining the wave function and energy states of an atom. These are determined from Schrödinger's equation. In the case of the hydrogen atom, it is possible to find wave functions ψ that are functions only of r, the distance from the proton to the electron. Such functions are called **spherically symmetric** and satisfy the simpler equation

$$\text{(1)} \qquad \frac{1}{r}\frac{d^2}{dr^2}(r\psi) = \frac{-8m\pi^2}{h^2}\left(E + \frac{e_0^2}{r}\right)\psi \ ,$$

where e_0^2, m, and h are constants and E, also a constant, represents the energy of the atom, which we assume here to be negative.

(a) Show that with the substitutions

$$r = \frac{h^2}{4\pi^2 m e_0^2}\rho \ , \qquad E = \frac{2\pi^2 m e_0^4}{h^2}\varepsilon \ ,$$

where ε is a negative constant, equation (1) reduces to

$$\frac{d^2(\rho\psi)}{d\rho^2} = -\left(\varepsilon + \frac{2}{\rho}\right)\rho\psi \ .$$

(b) If $f := \rho\psi$, then the preceding equation becomes

$$\text{(2)} \qquad \frac{d^2 f}{d\rho^2} = -\left(\varepsilon + \frac{2}{\rho}\right)f \ .$$

Show that the substitution $f(\rho) = e^{-\alpha\rho}g(\rho)$, where α is a positive constant, transforms (2) into

$$\text{(3)} \qquad \frac{d^2 g}{d\rho^2} - 2\alpha\frac{dg}{d\rho} + \left(\frac{2}{\rho} + \varepsilon + \alpha^2\right)g = 0 \ .$$

(c) If we choose $\alpha^2 = -\varepsilon$ (ε negative), then (3) becomes

$$\text{(4)} \qquad \frac{d^2 g}{d\rho^2} - 2\alpha\frac{dg}{d\rho} + \frac{2}{\rho}g = 0 \ .$$

Show that a power series solution $g(\rho) = \sum_{k=1}^{\infty} a_k \rho^k$ (starting with $k = 1$) for (4) must have coefficients a_k that satisfy the recurrence relation

$$\text{(5)} \qquad a_{k+1} = \frac{2(\alpha k - 1)}{k(k+1)}a_k \ , \qquad k \geq 1 \ .$$

(d) Now for $a_1 = 1$ and k very large, $a_{k+1} \approx (2\alpha/k)a_k$ and so $a_{k+1} \approx (2\alpha)^k/k!$, which are the coefficients for $\rho e^{2\alpha\rho}$. Hence, g acts like $\rho e^{2\alpha\rho}$, so $f(\rho) = e^{-\alpha\rho}g(\rho)$ is like $\rho e^{\alpha\rho}$. Going back further, we then see that $\psi \approx e^{\alpha\rho}$. Therefore, when $r = h^2\rho/4\pi^2 m e_0^2$ is large, so is ψ. Roughly speaking, $\psi^2(r)$ is proportional to the probability of finding an electron a distance r from the proton. Thus, the above argument would imply that the electron in a hydrogen atom is more likely to be found at a very large distance from the proton! Since this makes no sense physically, we ask: Do there exist positive values for α for which ψ remains bounded as r becomes large?

Show that when $\alpha = 1/n$, $n = 1, 2, 3, \ldots$, then $g(\rho)$ is a polynomial of degree n and argue that ψ is therefore bounded.

(e) Let E_n and $\psi_n(\rho)$ denote, respectively, the energy state and wave function corresponding to $\alpha = 1/n$. Find E_n (in terms of the constants e_0^2, m, and h) and $\psi_n(\rho)$ for $n = 1$, 2, and 3.

C Airy's Equation

In aerodynamics one encounters the following initial value problem for **Airy's equation:**

$$y'' + xy = 0\ , \qquad y(0) = 1\ , \qquad y'(0) = 0\ .$$

(a) Find the first five nonzero terms in a power series expansion about $x = 0$ for the solution and graph this polynomial for $-10 \le x \le 10$.

(b) Using the Runge–Kutta subroutine (see Section 5.3) with $h = 0.05$, approximate the solution on the interval [0, 10], i.e., at the points 0.05, 0.1, 0.15, etc.

(c) Using the Runge–Kutta subroutine with $h = 0.05$, approximate the solution on the interval $[-10, 0]$. [*Hint:* With the change of variables $z = -x$, it suffices to approximate the solution to $y'' - zy = 0$; $y(0) = 1$, $y'(0) = 0$, on the interval $[0, 10]$.]

(d) Using your knowledge of constant-coefficient equations as a basis for guessing the behavior of the solutions to Airy's equation, decide whether the power series approximation obtained in part (a) or the numerical approximation obtained in parts (b) and (c) better describes the true behavior of the solution on the interval $[-10, 10]$.

D Buckling of a Tower

A tower is constructed of four angle beams connected by diagonals (see Figure 8.16 on page 496). The deflection curve $y(x)$ for the tower is governed by the equation

$$x^2\frac{d^2y}{dx^2} + \frac{Pa^2}{EI}y = 0\ , \qquad a < x < a + L\ , \tag{6}$$

where x is the vertical coordinate measured down from the extended top of the tower, y is the deflection from the vertical passing through the center of the unbuckled tower, L is the tower height, a is the length of the truncation, P is the load, E is the modulus of elasticity, and I is the moment of inertia. The appropriate boundary conditions for this design are

$$y(a) = 0\ , \tag{7}$$

$$y'(a + L) = 0\ . \tag{8}$$