Mathematics 215 Solutions to Midterm #2

1. Let $x_1 = x, x_2 = y$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{g}(t) = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{\underline{A}} = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}$. Then the given nonhomogeneous linear system of differential equations becomes

$$\mathbf{x}' = \underline{\underline{A}}\mathbf{x} + \mathbf{g}(t) \quad (1).$$

First we find the general solution of the homogeneous system

$$\mathbf{x}' = \underline{\underline{A}}\mathbf{x}. \quad (2)$$

Then $\mathbf{x} = \boldsymbol{\xi} e^{\lambda t}$ solves (2) iff $\underline{\underline{A}}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$. For a nontrivial solution, it is necessary that $p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (2-\lambda)(-3-\lambda) + 4 = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) = 0$. Hence the matrix $\underline{\underline{A}}$ has eigenvalues $\lambda = \lambda_1 = 1, \lambda = \lambda_2 = -2$. Let $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix}$ be an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. Then $\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix} = 0$.

Hence, $\xi_2^{(1)} = \xi_1^{(1)}$, and, without loss of generality, one can choose $\xi_2^{(1)} = \xi_1^{(1)} = 1$ so that $\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{bmatrix}$ be an eigenvector corresponding to the eigenvalue $\lambda_2 = -2$. Then $\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{bmatrix} = 0$. Hence, $\xi_2^{(2)} = 4\xi_1^{(2)}$, and, without loss of generality, one can choose $\xi_1^{(2)} = 1$ so that $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Consequently, the general solution of (2) is given by $\mathbf{x} = \mathbf{x}_h = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Now, we seek a particular solution of (1) of the form $\mathbf{x} = \mathbf{x}_p = \boldsymbol{\xi} e^t + \boldsymbol{\eta} t e^t$ (3).

Then $\mathbf{x}'_p = \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t + \boldsymbol{\eta} t e^t$, $\underline{\underline{A}}\mathbf{x}_p = \underline{\underline{A}}\boldsymbol{\xi} e^t + \underline{\underline{A}}\boldsymbol{\eta} t e^t$. Thus (3) solves (1) iff

$$\underline{\underline{A}}\boldsymbol{\eta} = \boldsymbol{\eta}, \qquad (4)$$

$$(\underline{\underline{A}} - \underline{\underline{I}})\boldsymbol{\xi} = \boldsymbol{\eta} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (5)$$

From (4), it follows that $\boldsymbol{\eta} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with α to be determined so that (5) has a solution. Let $\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ Then (5) becomes

$$\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha - 1 \end{bmatrix} \quad (6)$$

But (6) has a solution iff $\alpha - 1 = 4\alpha \leftrightarrow \alpha = -\frac{1}{3}$.Then $\xi_1 - \xi_2 = -\frac{1}{3}$ (7). Without loss of generality, a particular solution of (7) is given by $\xi_2 = 0, \xi_1 = -\frac{1}{3}$. Thus $\mathbf{x}_p = -\frac{1}{3}\begin{bmatrix}1\\0\end{bmatrix}e^t - \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}te^t$, and hence the general solution of (1) is given by $\mathbf{x} = -\frac{1}{3}\begin{bmatrix}1\\0\end{bmatrix}e^t - \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}te^t + c_1e^t\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{-2t}\begin{bmatrix}1\\4\end{bmatrix}$. From the given initial conditions, one has $x_1(0) = -\frac{1}{3} + c_1 + c_2 = 1, \quad x_2(0) = c_1 + 4c_2 = 0 \Rightarrow c_1 = \frac{16}{9}, c_2 = -\frac{4}{9}$. Thus the solution of the given IVP is given by

$x = -\frac{1}{3}te^t + \frac{13}{9}e^t - \frac{4}{9}e^{-2t}, \quad y = -\frac{1}{3}te^t + \frac{16}{9}e^t - \frac{16}{9}e^{-2t}$.

2. (a) $f(t) = u_0(t) - 2u_h(t) + u_{2h}(t)$. Let $F(s)$ be the Laplace transform of $f(t)$. Then $F(s) = \frac{1}{s}[1 - 2e^{-hs} + e^{-2hs}]$.

(b) Let $Y(s)$ be the Laplace transform of $y(t)$. Then $(s^2+1)Y(s) = F(s)$, i.e.,

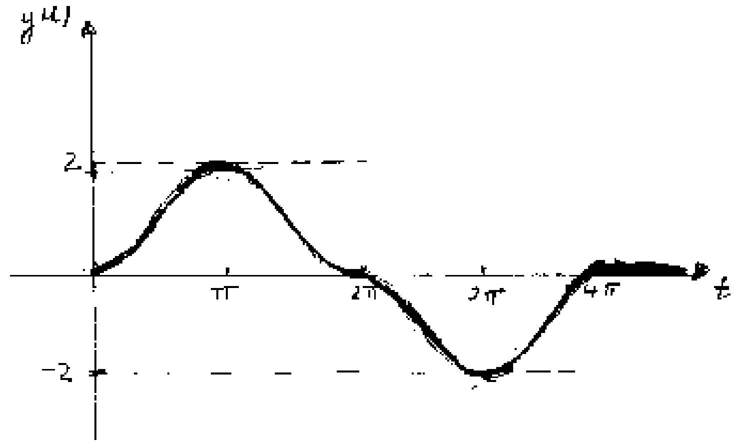
$Y(s) = \frac{1}{s(s^2+1)}[1 - 2e^{-hs} + e^{-2hs}]$. Now use partial fractions. Then $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1}$ for some particular constants A, B, C. Letting $s \to 0$, one immediately sees that A = 1. Letting $s \to \infty$, one sees that $B + A = 0 \Rightarrow B = -1$. Since $\frac{1}{s(s^2+1)}$ is an odd function of s, it then follows that C = 0. Thus $Y(s) = \left[\frac{1}{s} - \frac{s}{s^2+1}\right][1 - 2e^{-hs} + e^{-2hs}]$. Since the Laplace transform of $\cos t$ is $\frac{s}{s^2+1}$, it follows that the solution of the initial value problem is given by $y(t) = f(t) - \cos t + 2u_h(t)\cos(t-h) - u_{2h}(t)\cos(t-2h)$.

(c) If $h = 2\pi$, then

$$y(t) = \begin{cases} 1 - \cos t, & 0 \le t < 2\pi \\ -1 + \cos t, & 2\pi \le t < 4\pi \\ 0, & t \ge 4\pi \end{cases}$$

Note, of course, that the solution and its first derivative are both continuous. A sketch of the solution is given on the next page.

(c)



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