MATH 255 – Midterm I

October 16, 2006

Your full name: ______________________

ID Number: ______________________

Scores:

Problem 1 ______________ (5 points)

Problem 2 ______________ (5 points)

Problem 3 ______________ (5 points)

Problem 4 ______________ (5 points)

Problem 5 ______________ (5 points)

TOTAL: ______________ **(25 points)**

One 8.5×11 sheet of notes allowed.
Show all your work and make your reasoning clear.

Problem 1 (5 points)
Find the solution of

$$y' = x + 2xy, \qquad y(0) = 1.$$

Standardform: $y' - 2xy = x, \; y(0) = 1$
(linear, non-constant coefficients)

integrating factor $\mu' = -2x\mu \Rightarrow \underline{\underline{\mu = e^{-x^2}}}$

$$\Rightarrow \quad e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2}$$

$$(e^{-x^2}y)' = xe^{-x^2}$$

$$\Rightarrow \quad e^{-x^2}y = \int xe^{-x^2}dx + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$y = -\frac{1}{2} + Ce^{+x^2}$$

$$y(0) = -\frac{1}{2} + C \overset{!}{=} 1 \Rightarrow C = \frac{3}{2}$$

$$\underline{\underline{y = -\frac{1}{2} + \frac{3}{2}e^{x^2}}}$$

Problem 2 (5 points)
Find the general solution of

$$y'' - 5y' + 6y = 4xe^x$$

particular solution:

$$Y = (Ax+B)e^x$$

$$Y' = (A+B)e^x + Axe^x$$

$$Y'' = (2A+B)e^x + Axe^x$$

$$Y'' - 5Y' + 6Y = (2A+B)e^x + Axe^x - 5\left((A+B)e^x + Axe^x\right) + 6\left(Be^x + Axe^x\right) \stackrel{!}{=} 4xe^x$$

$$\Rightarrow (2A - 5A + B - 5B + 6B)e^x + 2Axe^x \stackrel{!}{=} 4xe^x$$

$$\Rightarrow 2A = 4 \Rightarrow \underline{\underline{A = 2}} \quad \text{and} \quad -3A + 2B = 0$$

$$\Rightarrow \underline{\underline{B = 3}}$$

general solution

$$\underline{\underline{y(t) = c_1 e^{2x} + c_2 e^{3x} + (2x+3)e^x}}$$

Problem 3 (5 points)
Find the general solution of

$$y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Hint: Use the substitution $z = \frac{y}{x}$.

$$xz = y \quad \Rightarrow y' = z + xz'$$

$$\Rightarrow z + xz' = 1 + z + z^2$$

$$\Rightarrow z' = \frac{1}{x}(1 + z^2) \quad \text{separable}$$

$$\frac{dz}{1+z^2} = \frac{dx}{x}$$

$$\tan^{-1}(z) = \log|x| + c$$

$$z = \tan(\log|x| + c)$$

$$\Rightarrow \underline{\underline{y = x\tan(\log|x| + c)}}$$

Problem 4 (5 points)

Consider a cylindrical tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole of area a in the bottom. Let $h = h(t)$ be the depth of water in the tank at time t. Due to Torricelli's principle, it satisfies the differential equation

$$Ah' = (k - \alpha a\sqrt{2gh}),$$

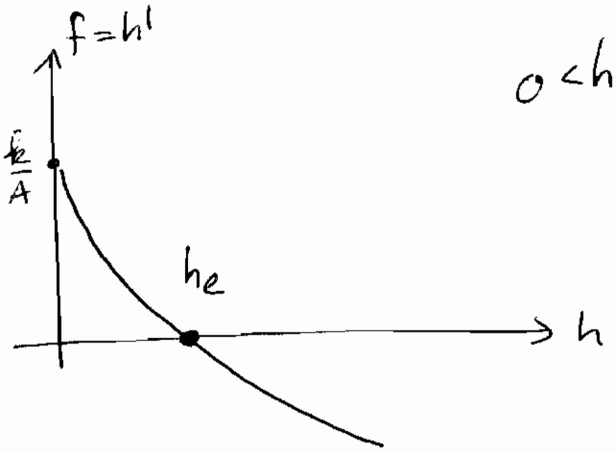
where g is the acceleration due to gravity, and α is a contraction coefficient with $0.5 \le \alpha \le 1$. Determine the equilibrium depth h_e of water and discuss the asymptotic stability of the equilibrium solution.

Equilibrium solution: $k - \alpha a\sqrt{2gh} \overset{!}{=} 0$

$\Rightarrow k^2 = \alpha^2 a^2 \, 2gh$

$\Rightarrow \underline{\underline{h_e = \dfrac{k^2}{2\alpha^2 a^2 g}}}$

The graph of $f(h) = \dfrac{k - \alpha a\sqrt{2gh}}{A}$ looks like



$0 < h < h_e$: $f(h)$ positive $\Rightarrow h$ increasing

$h > h_e$: $f(h)$ negative $\Rightarrow h$ decreasing

$\Rightarrow$ <u>h_e is asymptotically stable</u>

Problem 5 (5 points)
Consider

$$y'' - \frac{1}{2x}y' + \frac{1}{2x^2}y = 0, \qquad x \geq 1.$$

Show that the functions $y_1(x) = x$ and $y_2(x) = \sqrt{x}$ solve the equation and form a fundamental set of solutions.

$\underline{y_1}$: $y_1' = 1$, $y_1'' = 0 \Rightarrow 0 - \frac{1}{2x} \cdot 1 + \frac{1}{2x^2} \cdot x = \underline{\underline{0}}$ ✓

$\underline{y_2}$: $y_2' = \frac{1}{2} x^{-1/2}$, $y_2'' = -\frac{1}{4} x^{-3/2} \Rightarrow -\frac{1}{4} x^{-3/2} - \frac{1}{2x} \frac{1}{2} x^{-1/2} + \frac{1}{2x^2} x^{1/2}$

$= -\frac{2}{4} x^{-3/2} + \frac{1}{2} x^{-3/2} = \underline{\underline{0}}$ ✓

<u>Wronskian</u>.

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \Big|_x = \det \begin{pmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2} x^{-1/2} \end{pmatrix} = \frac{\sqrt{x}}{2} - \sqrt{x}$$

$$= \underline{\underline{-\frac{\sqrt{x}}{2}}} \neq 0 \quad (\text{since } x \geq 1)$$

$\Rightarrow$ <u>y_1 and y_2 are a fundamental set of solutions</u>