# MATH 255 – Midterm II

November 15, 2006

**Your full name:** ______________________

**ID Number:** ______________________

**Scores:**

Problem 1 ____________ (5 points)

Problem 2 ____________ (5 points)

Problem 3 ____________ (5 points)

Problem 4 ____________ (5 points)

**TOTAL:** ____________ **(20 points)**

One $8.5 \times 11$ sheet of notes allowed.
**Show all your work and make your reasoning clear.**

**Problem 1 (5 points)**

Find the general solution of

$$y' = 2y + (4 - x).$$

First-order & linear $\Rightarrow$ method of integrating factor

Standard form: $y' - 2y = 4 - x$

Integrating factor: $\mu'(x) = -2\mu(x) \Rightarrow \mu(x) = e^{-2x}$

$$\Rightarrow \left(e^{-2x} y\right)' = e^{-2x}(4 - x)$$

$$\Rightarrow e^{-2x} y = 4\int e^{-2x}\,dx - \int x e^{-2x}\,dx + C$$

$$= -2e^{-2x} - \left(\frac{1}{-2} x e^{-2x} - \int \frac{1}{-2} e^{-2x}\,dx\right) + C$$

$$= -2e^{-2x} + \frac{1}{2} x e^{-2x} + \frac{1}{4} e^{-2x} + C$$

$$\Rightarrow y = -2 + \frac{x}{2} + \frac{1}{4} + Ce^{2x}$$

$$\underline{\underline{y = Ce^{2x} + \frac{x}{2} - \frac{7}{4}}}$$

**Problem 2 (5 points)**

Find the general solution of

$$x' = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} x.$$

characteristic equation:

$$\det \begin{pmatrix} 1-r & 2 \\ 3 & 6-r \end{pmatrix} = (1-r)(6-r) - 6$$
$$= r^2 - 7r = r(r-7) \overset{!}{=} 0$$

$\Rightarrow$ eigenvalues: $r_1 = 0$, $r_2 = 7$

eigenvectors:

$r = 0$: $\xi_1 + 2\xi_2 = 0 \Rightarrow \xi^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$r = 7$: $-6\xi_1 + 2\xi_2 = 0 \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\Rightarrow \underline{\underline{x(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{7t}}}$$

**Problem 3 (5 points)**

Use the Laplace transform to solve

$$y'' - 3y' + 2y = \begin{cases} 0 & 0 \le t < 3, \\ 1 & 3 \le t < 6, \\ 0 & t \ge 6, \end{cases} \qquad y(0) = 0,\ y'(0) = 0$$

step function : $g(t) = u_3(t) - u_6(t)$

transform $\quad (s^2 - 3s + 2)\,Y(s) = \mathcal{L}\{u_3\} - \mathcal{L}\{u_6\} = \dfrac{e^{-3s}}{s} - \dfrac{e^{-6s}}{s}$

$\Rightarrow Y(s) = \left(e^{-3s} - e^{-6s}\right) \dfrac{1}{s(s-2)(s-1)} = \left(e^{-3s} - e^{-6s}\right) H(s)$

partial fraction decomposition

$$H(s) = \frac{1}{2s} + \frac{1}{2}\,\frac{1}{s-2} - \frac{1}{s-1}$$

backtransform :

$$\underline{\underline{y(t)}} = \mathcal{L}^{-1}\{e^{-3s} H(s)\} - \mathcal{L}^{-1}\{e^{-6s} H(s)\}$$

$$= \underline{\underline{u_3(t)\, h(t-3) - u_6(t)\, h(t-6)}}$$

where $\underline{\underline{h}} = \mathcal{L}^{-1}\{H\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$$= \underline{\underline{\frac{1}{2} + \frac{1}{2}e^{2t} - e^{t}}}$$

**Problem 4 (5 points)**
Consider the 2-by-2 system $x' = Ax$ where the matrix $A$ is skew-symmetric, i.e. $A = -A^\top$. Show that $x_1^2 + x_2^2$ is constant in time. (Hint: Show $\frac{d}{dt}(x_1^2 + x_2^2) = 0$)

$A$ skew-symmetric : $A = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$ . for a constant $k$

$\Rightarrow$ the system is

$$x_1' = k x_2$$
$$x_2' = -k x_1$$

$$\Rightarrow \frac{d}{dt}(x_1^2 + x_2^2) = 2 x_1 x_1' + 2 x_2 x_2'$$
$$= 2 x_1 k x_2 + 2 x_2 (-k x_1) = 0$$

$\Rightarrow x_1^2 + x_2^2$ is constant in time.