Math 215/255 Midterm Exam I

Oct 4, 2010

Duration: 50 minutes

Last Name: ____________ First Name: ____________ Student Number: ________

Do not open this test until instructed to do so! This exam should have 6 pages, including this cover sheet. It is a closed book exam; no textbooks, calculators, laptops, formula sheets or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Circle your solutions! Reduce your answer as much as possible. Explain your work. Relax.** Use the extra pages if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	Out of	Score
1	20	
2	10	
3	20	
Total	50	

Problem 1 (20 points)

Solve the following initial value problems for $y(x)$:

(a) $2y'' - y' - y = 0$; $y(0) = 3$, $y'(0) = 0$.

characteristic equation: $2r^2 - r - 1 = (2r+1)(r-1) = 0$

roots: $r_1 = -1/2$, $r_2 = 1$

general solution: $y(t) = c_1 e^{-1/2x} + c_2 e^x$

$y'(t) = -\frac{c_1}{2} e^{-1/2x} + c_2 e^x$

initial conditions:

$$\left.\begin{array}{l} y(0) = 3 \Rightarrow c_1 + c_2 = 3 \\ y'(0) = 0 \Rightarrow -\frac{c_1}{2} + c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 2c_2 \\ c_2 = 1, \; c_1 = 2 \end{array}$$

solution:

$$\underline{y(x) = 2e^{-\frac{1}{2}x} + e^x}$$

(b) $y' = (x+y)^2 - 1$; $y(0) = 1$.

Hint: use the substitution $z = x + y$.

$y = z - x \Rightarrow y' = z' - 1$

$\Rightarrow z' - 1 = z^2 - 1 \Rightarrow z' = z^2$ separable

$\Rightarrow \frac{dz}{z^2} = dx$ or $-\frac{1}{z} = x + c$

$\Rightarrow z = -\frac{1}{x+c}$

$\Rightarrow y = -\frac{1}{x+c} - x = -\frac{1 + x(x+c)}{x+c}$

Initial condition: $y(0) = -\frac{1}{c} = 1 \Rightarrow \underline{c = -1}$

Hence,

$$\underline{y(x) = -\frac{x^2 - x + 1}{x - 1} = \frac{x^2 - x + 1}{1 - x}}$$

Problem 2 (10 points)

Consider the initial value problem

$$(1-x)\frac{dy}{dx} - y = \frac{1}{x+1}, \tag{1}$$

$$y(0) = 1. \tag{2}$$

(a) Over what interval in x does a unique solution to this problem exist?

standard form $\frac{dy}{dX} - \frac{y}{1-X} = \frac{1}{(X+1)(1-X)} = \frac{dy}{dX} + p(X)\,y = g(X)$

with $p(X) = \frac{1}{X-1}$ continuous except for $X = 1$

$g(X) = \frac{1}{(X+1)(1-X)}$ continuous except for $X = \pm 1$

$\Rightarrow$ unique solution exists for $-1 < X < 1$

(b) Find the solution to this initial value problem.

Integrating factor: $\mu' = p(X)\mu \Rightarrow \mu(X) = e^{\int p(X)\,dX}$

part a)

So: $\mu(X) = e^{\ln|X-1|} = e^{\ln(1-X)} = 1-X$

Then: $(\mu y)' = g(X)\mu(X) = \frac{1}{X+1}$

$\Leftrightarrow \left((1-X)y\right)' = \frac{1}{X+1}$

part a)

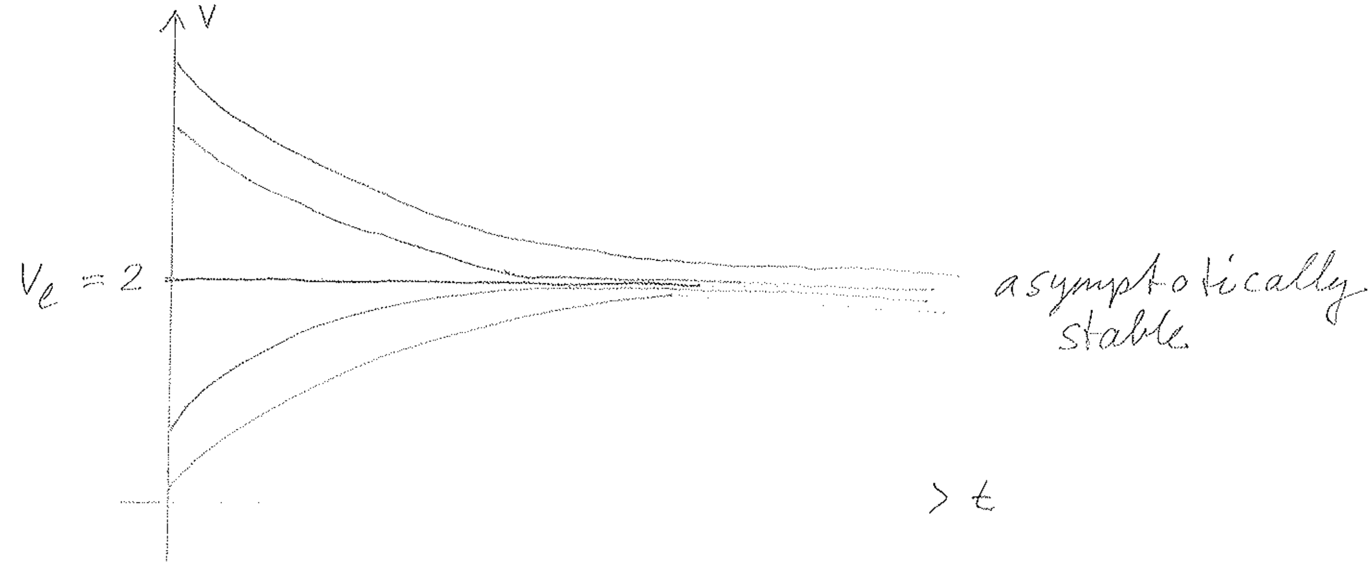
$\Leftrightarrow (1-X)y = \int \frac{1}{X+1} + C = \ln|X+1| + C = \ln(X+1) + C$

$X = 0$: $C = 1$

$\Rightarrow y(X) = \frac{\ln(X+1)+1}{1-X}$

Problem 3 continued

(b) Sketch typical solution curves for different values of the initial velocity v_0.



(c) If the initial velocity is v_0=4m/s, at what time will the bead's velocity be 3m/s?

$$V(t) = 2 + 2e^{-5t} \overset{!}{=} 3$$

$$\Rightarrow e^{-5t} = 1/2$$

$$\Rightarrow -5t = \ln(1/2) = -\ln(2)$$

$$\Rightarrow \underline{\underline{t = \frac{\ln(2)}{5}}}$$

Problem 3 (20 points)

A bead of mass m is dropped into a large barrel of oil. We assume that the bead is acted upon by a constant gravitational force and by an opposing drag force proportional to its velocity, due to the oil's viscosity. The bead's downward speed $v(t)$ can be described by the equation

$$mv'(t) + \gamma v(t) = mg,$$

where γ is the drag coefficient and g represents acceleration due to gravity. Assume $m = 0.2\text{kg}$, $\gamma = 1\text{kg/s}$, and $g = 10\ \text{m/s}^2$.

(a) Find a general expression for the bead's velocity if its initial velocity is $v(0) = v_0$.

$0.2\,v' + v = 2$ first-order, linear, constant coefficients

or: $v' + 5v = 10$

the solution is

$\underline{v(t) = 2 + (v_0 - 2)e^{-5t}}$

(equilibrium solution is $v = 2$)

PROBLEM CONTINUES ON NEXT PAGE