Math 215/255 Midterm Exam II

Oct 25, 2010

Duration: 50 minutes

Last Name: ____________ First Name: ____________ Student Number: ________

Circle the course in which you are registered: Math 215 or Math 255

Do not open this test until instructed to do so! This exam should have 7 pages, including this cover sheet. It is a closed book exam; no textbooks, calculators, laptops, formula sheets or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam. **Circle your solutions! Reduce your answer as much as possible. Explain your work. Relax.** Use the extra pages if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	Out of	Score
1	20	
2	10	
3	10	
Total	40	

Problem 1 (20 points)

Consider the initial value problem for $y(x)$:

$$y'' + 8y' + ky = 0,\ y(0) = 0,\ y'(0) = 6$$

where k is a constant.
(a) Find the solution if $k = 7$. Describe the behaviour of the solution.

$y'' + 8y' + 7y = 0 \Rightarrow$ char. equation: $r^2 + 8r + 7 = 0$

$\Rightarrow (r+7)(r+1) = 0 \Rightarrow$ roots: $r_1 = -7,\ r_2 = -1$

general solution: $y(x) = c_1 e^{-7x} + c_2 e^{-x}$

$y'(x) = -7c_1 e^{-7x} - c_2 e^{-x}$

initial condition: $c_1 + c_2 = 0$ & $-7c_1 - c_2 = 6$

$\Rightarrow c_1 = -1;\ c_2 = +1$

$\Rightarrow \underline{y(x) = e^{-x} - e^{-7x}}$ exponential decay. (overdamped case)

(b) Find the solution if $k = 25$. Describe the behaviour of the solution.

$y'' + 8y' + 25y = 0 \Rightarrow$ char. equation: $r^2 + 8r + 25 = 0$

roots: $r_{1,2} = \dfrac{-8 \pm \sqrt{64 - 4\cdot 25}}{2} = \dfrac{-8 \pm 6i}{2} = -4 \pm 3i$

general solution: $y(x) = c_1 e^{-4x}\sin(3x) + c_2 e^{-4x}\cos(3x)$

$y(0) = 0 + c_2 \overset{!}{=} 0 \Rightarrow \underline{c_2 = 0}$

$y'(0) = \left(-4c_1 e^{-4x}\sin(3x) + c_1 e^{-4x}\, 3\cdot\cos(3x)\right)\Big|_{x=0} \overset{!}{=} 6$

$\Rightarrow \underline{c_1 = 2}$

$\Rightarrow y(x) = 2e^{-4x}\sin(3x)$ exponential decay and oscillations (damped oscillations)

Problem 1 (continued)

(c) These solutions exhibit different behaviours. Find the critical value k_c of k where the nature of the solution changes.

$y'' + 8y' + ky = 0 \Rightarrow$ char. equation: $r^2 + 8r + k = 0$

nature of solution changes for

$$64 - 4k = 0 \Rightarrow \underline{\underline{k_c = 16}}$$

(d) Find the solution if $k = k_c$.

We have now the double root $r_1 = r_2 = -4$.

Hence, the general solution is

$$y(x) = (C_1 + C_2 x)e^{-4x}$$

$$y(0) = 0 \Rightarrow C_1 \overset{!}{=} 0$$

$$y'(0) = \left(C_2 e^{-4x} + C_2 x(-4)e^{-4x}\right)\Big|_{x=0} \overset{!}{=} 6$$

$$\Rightarrow C_2 = 6$$

$$\Rightarrow \underline{\underline{y(x) = 6xe^{-4x}}}$$

(critical damping)

Problem 2 (10 points)
Use variation of parameters to find the general solution of

$$xy'' + (5x-1)y' - 5y = x^2e^{-5x},\ x > 0,$$

given that two linearly independent solutions to the homogeneous equation are $y_1(x) = 5x-1$ and $y_2(x) = e^{-5x}$.

Wronskian: $W(x) = \begin{vmatrix} 5x-1 & e^{-5x} \\ 5 & -5e^{-5x} \end{vmatrix}$

$$= (5x-1)(-5)e^{-5x} - 5e^{-5x} = \underline{-25xe^{-5x}}$$

Dividing the equation by x ($\Rightarrow$ standard form), a particular solution is given by $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, with

$$\underline{u_1(x)} = -\int \frac{y_2(x)\, x e^{-5x}}{W(x)}\,dx = -\int \frac{e^{-5x}\, x e^{-5x}}{-25xe^{-5x}}\,dx$$

$$= \frac{1}{25}\int e^{-5x}\,dx = \underline{-\frac{1}{125}e^{-5x}}$$

and

$$\underline{u_2(x)} = \int \frac{y_1(x)\, x e^{-5x}}{W(x)}\,dx = \int \frac{(5x-1)\, x e^{-5x}}{-25xe^{-5x}}\,dx$$

$$= -\frac{1}{25}\int (5x-1)\,dx = -\frac{1}{25}\left(\frac{5}{2}x^2 - x\right) = \underline{-\frac{x^2}{10} + \frac{x}{25}}$$

So the general solution is

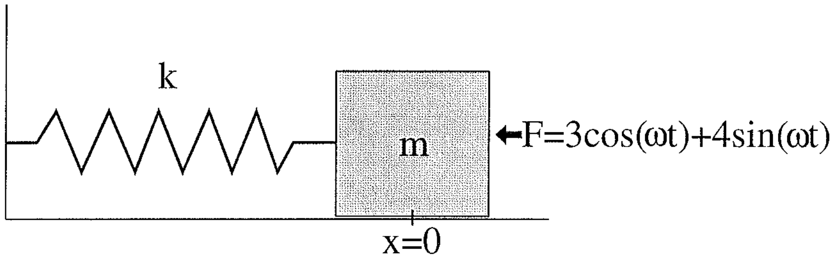
$$\underline{y(x)} = A(5x-1) + Be^{-5x} - \frac{1}{125}e^{-5x}(5x-1) + \left(\frac{x}{25} - \frac{x^2}{10}\right)e^{-5x}$$

$$= A(5x-1) + Be^{-5x} + \underbrace{\frac{1}{125}e^{-5x}}_{\text{can be absorbed in constant}} - \frac{x^2}{10}e^{-5x}$$

$$= \underline{C_1(5x-1) + C_2e^{-5x} - \frac{x^2}{10}e^{-5x}}$$

Problem 3 (10 points)

Consider a mass $m = 1$kg tethered to the wall by a spring with spring constant $k = 4$kg·m/s, sitting at rest on a frictionless surface (see figure below). A forcing $F(t) = 3\cos(\omega t) + 4\sin(\omega t)$ ($\omega \neq 2$) is then applied to the spring.



The spring's motion is described by:

$$x''(t) + 4x(t) = 3\cos(\omega t) + 4\sin(\omega t),$$

where $x(t)$ is the distance of the mass from its equilibrium position (see figure).

(a) Find the *particular solution* of the above equation for the position of the mass at time t. You do *not* need to calculate the homogeneous solution.

$$X(t) = A\cos(wt) + B\sin(wt)$$

$$X'(t) = -Aw\sin(wt) + Bw\cos(wt)$$

$$X''(t) = -Aw^2\cos(wt) - Bw^2\sin(wt)$$

$$\Rightarrow X'' + 4X = -Aw^2\cos(wt) - Bw^2\sin(wt) + 4A\cos(wt) + 4B\sin(wt)$$

$$= \cos(wt)A(4-w^2) + \sin(wt)B(4-w^2) \stackrel{!}{=} 3\cos(wt) + 4\sin(wt)$$

$$\Rightarrow A = \frac{3}{4-w^2} \quad \text{and} \quad B = \frac{4}{4-w^2}$$

$$X(t) = \frac{3\cos(wt) + 4\sin(wt)}{4-w^2}$$

Problem 3 (continued)

(b) The particular solution can be written in the form $x_p(t) = R(\omega)\cos(\omega t - \delta)$. Find the amplitude $R(\omega)$.

$$R^2 = A^2 + B^2 = \frac{3^2 + 4^2}{(4-w^2)^2} = \frac{25}{(4-w^2)^2}$$

$$\Rightarrow \underline{R = \frac{5}{|4-w^2|}}$$

(c) Plot $R(\omega)$ versus ω. Label the intercepts and asymptotes of the graph. What is the significance of the asymptote?

vertical asymptote for $w = 2$: resonance