Discontinuous Forcing Functions Example

Problem: Consider an idealized LCR circuit with no resistance such that its natural frequency $1/\sqrt{CL} = \beta$. Assume that initially there is no charge or current in the circuit. From time $t = \pi$ to time $t = 2\pi$ we impose some forcing at some frequency $\omega \neq \beta$ of the form $\sin(\omega t)$. The initial value problem for the charge $q(t)$ is therefore:

$$q'' + \beta^2 q = \begin{cases} 0, & 0 \leq t < \pi \\ \sin(\omega t), & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}, \quad q(0) = q'(0) = 0.$$

What is the charge in the circuit $q(t)$ at time t?

Solution:

We'll solve using Laplace transforms. We begin by writing the RHS using unit step functions.

$$\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right) = \begin{cases} 0, & 0 \leq t < \pi \\ \sin(\omega t), & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}.$$

Our IVP can therefore be re-written as $q'' + \beta^2 q = \sin(\omega t)u_\pi(t) - \sin(\omega t)u_{2\pi}(t)$, $q(0) = q'(0) = 0$. Now we take the Laplace Transform of both sides of the equation. Let $Q(s) = \mathfrak{L}\{q(t)\}$. Then

$$\begin{aligned} \mathfrak{L}\{q'' + \beta^2 q\} &= \mathfrak{L}\{\sin(\omega t)u_\pi(t) - \sin(\omega t)u_{2\pi}(t)\} \\ \mathfrak{L}\{q''\} + \beta^2 \mathfrak{L}\{q\} &= \mathfrak{L}\{\sin(\omega t)u_\pi(t)\} - \mathfrak{L}\{\sin(\omega t)u_{2\pi}(t)\}, \end{aligned}$$

since $\mathfrak{L}$ is a linear operator. Going term-by-term:

- $\mathfrak{L}\{q''\} = s^2 Q(s) - sq(0) - q'(0) = s^2 Q(s)$ since $q(0) = q'(0) = 0$.
- $\beta^2 \mathfrak{L}\{q\} = \beta^2 Q(s)$
- Since $\mathfrak{L}\{f(t)u_c(t)\} = e^{-cs}\mathfrak{L}\{f(t+c)\}$,
  $\mathfrak{L}\{\sin(\omega t)u_\pi(t)\} = e^{-\pi s}\mathfrak{L}\{\sin(\omega(t+\pi))\} = e^{-\pi s}\mathfrak{L}\{\sin(\omega(t+\pi))\} = e^{-\pi s}\mathfrak{L}\{\sin(\omega t)\cos(\omega\pi) + \sin(\omega\pi)\cos(\omega t)\} = e^{-\pi s}\cos(\omega\pi)\mathfrak{L}\{\sin(\omega t)\} + e^{-\pi s}\sin(\omega\pi)\mathfrak{L}\{\cos(\omega t)\}$

$$\Rightarrow \mathfrak{L}\{\sin(\omega t)u_\pi(t)\} = \cos(\omega\pi)\frac{\omega e^{-\pi s}}{s^2 + \omega^2} + \sin(\omega\pi)\frac{s e^{-\pi s}}{s^2 + \omega^2}.$$

- Similarly we find $\mathfrak{L}\{\sin(\omega t)u_{2\pi}(t)\}$

$$\mathfrak{L}\{\sin(\omega t)u_\pi(t)\} = \cos(2\omega\pi)\frac{\omega e^{-2\pi s}}{s^2 + \omega^2} + \sin(2\omega\pi)\frac{s e^{-2\pi s}}{s^2 + \omega^2}.$$

Therefore the transformed IVP $q'' + \beta^2 q = \sin(\omega t)u_\pi(t) - \sin(\omega t)u_{2\pi}(t)$, $q(0) = q'(0) = 0$ is

$$s^2 Q(s) + \beta^2 Q(s) = \cos(\omega\pi)\frac{\omega e^{-\pi s}}{s^2 + \omega^2} + \sin(\omega\pi)\frac{s e^{-\pi s}}{s^2 + \omega^2} - \cos(2\omega\pi)\frac{\omega e^{-2\pi s}}{s^2 + \omega^2} - \sin(2\omega\pi)\frac{s e^{-2\pi s}}{s^2 + \omega^2}.$$

Solving for $Q(s)$ we find

$$\begin{aligned} Q(s) = {} & \cos(\omega\pi)\frac{\omega e^{-\pi s}}{(s^2 + \omega^2)(s^2 + \beta^2)} + \sin(\omega\pi)\frac{s e^{-\pi s}}{(s^2 + \omega^2)(s^2 + \beta^2)} \\ & - \cos(2\omega\pi)\frac{\omega e^{-2\pi s}}{(s^2 + \omega^2)(s^2 + \beta^2)} - \sin(2\omega\pi)\frac{s e^{-2\pi s}}{(s^2 + \omega^2)(s^2 + \beta^2)}. \end{aligned}$$

To find the charge $q(t)$ all that's left to do is invert the transform, $q(t) = \mathfrak{L}^{-1}\{Q(s)\}$. This is the most involved part! Use partial fraction decompositions to split this up into bits we recognize from the table. We use

$$\begin{aligned}\frac{1}{(s^2+\omega^2)(s^2+\beta^2)} &= \frac{A}{(s^2+\omega^2)} + \frac{B}{(s^2+\beta^2)} \\ \frac{s}{(s^2+\omega^2)(s^2+\beta^2)} &= \frac{Cs}{(s^2+\omega^2)} + \frac{Ds}{(s^2+\beta^2)}\end{aligned}$$

since we can find $\mathfrak{L}^{-1}\left\{a/(s^2+a^2)\right\}$ and $\mathfrak{L}^{-1}\left\{s/(s^2+a^2)\right\}$ in the table (a can be β or ω). We find $A = C = 1/(\beta^2 - \omega^2)$, $B = D = -1/(\beta^2 - \omega^2)$. Thus

$$\begin{aligned}Q(s) &= \frac{1}{(\beta^2-\omega^2)}\left[\cos(\omega\pi)\left(\frac{\omega e^{-\pi s}}{s^2+\omega^2} - \frac{\omega e^{-\pi s}}{s^2+\beta^2}\right) + \sin(\omega\pi)\left(\frac{s e^{-\pi s}}{s^2+\omega^2} - \frac{s e^{-\pi s}}{s^2+\beta^2}\right)\right. \\ &\quad \left. - \cos(2\omega\pi)\left(\frac{\omega e^{-2\pi s}}{s^2+\omega^2} - \frac{\omega e^{-2\pi s}}{s^2+\beta^2}\right) - \sin(2\omega\pi)\left(\frac{s e^{-2\pi s}}{s^2+\omega^2} - \frac{s e^{-2\pi s}}{s^2+\beta^2}\right)\right]\end{aligned}$$

(we could simplify further but we're leaving the pieces we recognize from the table alone!). Now we invert. We can go term-by-term on the RHS since $\mathfrak{L}^{-1}$ is linear. In summary,

- $\mathfrak{L}^{-1}\left\{1/(s^2+a^2)\right\} = \sin(at)/a$ and $\mathfrak{L}^{-1}\left\{s/(s^2+a^2)\right\} = \cos(at)$ where $a = \omega$ or β.
- Since $\mathfrak{L}^{-1}\left\{1/(s^2+a^2)\right\} = \sin(at)/a$ and $\mathfrak{L}^{-1}\left\{F(s)e^{-cs}\right\} = f(t-c)u_c(t)$ where $F(s) = \mathfrak{L}\{f(t)\}$, $\mathfrak{L}^{-1}\left\{e^{-n\pi s}/(s^2+a^2)\right\} = \sin(a(t-n\pi))u_{n\pi}(t)/a$, where a can be ω or β, n can be 1 or 3.
- Since $\mathfrak{L}^{-1}\left\{s/(s^2+a^2)\right\} = \cos(at)$ and $\mathfrak{L}^{-1}\left\{F(s)e^{-cs}\right\} = f(t-c)u_c(t)$ where $F(s) = \mathfrak{L}\{f(t)\}$, $\mathfrak{L}^{-1}\left\{se^{-n\pi s}/(s^2+a^2)\right\} = \cos(a(t-n\pi))u_{n\pi}(t)$, where a can be ω or β, n can be 1 or 3.

Using these inverse transforms,

$$\begin{aligned}\mathfrak{L}^{-1}\{Q(s)\} &= \mathfrak{L}^{-1}\left\{\frac{1}{(\beta^2-\omega^2)}\left[\cos(\omega\pi)\left(\frac{\omega e^{-\pi s}}{s^2+\omega^2} - \frac{\omega e^{-\pi s}}{s^2+\beta^2}\right) + \sin(\omega\pi)\left(\frac{s e^{-\pi s}}{s^2+\omega^2} - \frac{s e^{-\pi s}}{s^2+\beta^2}\right)\right.\right. \\ &\quad \left.\left. - \cos(2\omega\pi)\left(\frac{\omega e^{-2\pi s}}{s^2+\omega^2} - \frac{\omega e^{-2\pi s}}{s^2+\beta^2}\right) - \sin(2\omega\pi)\left(\frac{s e^{-2\pi s}}{s^2+\omega^2} - \frac{s e^{-2\pi s}}{s^2+\beta^2}\right)\right]\right\}\end{aligned}$$

becomes

$$\begin{aligned}q(t) &= \frac{1}{(\beta^2-\omega^2)}\left[\cos(\omega\pi)\left(\sin(\omega(t-\pi)) - \frac{\omega\sin(\beta(t-\pi))}{\beta}\right)u_\pi(t) + \sin(\omega\pi)\left(\cos(\omega(t-\pi)) - \cos(\beta(t-\pi))\right)u_\pi(t)\right. \\ &\quad \left. - \cos(2\omega\pi)\left(\sin(\omega(t-2\pi)) - \frac{\omega\sin(\beta(t-2\pi))}{\beta}\right)u_{2\pi}(t) - \sin(2\omega\pi)\left(\cos(\omega(t-2\pi)) - \cos(\beta(t-2\pi))\right)u_{2\pi}(t)\right].\end{aligned}$$

Simplifying we obtain:

$$\begin{aligned}q(t) &= \frac{1}{\beta(\beta^2-\omega^2)}\left[\beta\sin(\omega t) - \omega\cos(\omega\pi)\sin(\beta(t-\pi)) - \beta\sin(\omega\pi)\cos(\beta(t-\pi))\right]u_\pi(t) \\ &\quad - \frac{1}{\beta(\beta^2-\omega^2)}\left[\beta\sin(\omega t) - \omega\cos(2\omega\pi)\sin(\beta(t-2\pi)) - \beta\sin(2\omega\pi)\cos(\beta(t-2\pi))\right]u_{2\pi}(t)\end{aligned}$$

Alternatively, we could write $q(t)$ as:

$$q(t) = \begin{cases} 0, & 0 \le t < \pi \\ \frac{1}{\beta(\beta^2-\omega^2)}\left[\beta\sin(\omega t) - \omega\cos(\omega\pi)\sin(\beta(t-\pi)) - \beta\sin(\omega\pi)\cos(\beta(t-\pi))\right], & \pi \le t < 2\pi \\ \frac{1}{\beta(\beta^2-\omega^2)}\left[\beta\sin(\omega t) - \omega\cos(\omega\pi)\sin(\beta(t-\pi)) - \beta\sin(\omega\pi)\cos(\beta(t-\pi))\right] & t \ge 2\pi \\ \quad -\frac{1}{\beta(\beta^2-\omega^2)}\left[\beta\sin(\omega t) - \omega\cos(2\omega\pi)\sin(\beta(t-2\pi)) - \beta\sin(2\omega\pi)\cos(\beta(t-2\pi))\right], & \end{cases}$$

This is a very long and detailed problem, but the techniques are identical to those we used in class. There's more algebra and more to keep track of though - be careful! If you can solve problems like this, you're all set when it comes to IVPs with discontinuous forcing functions. FYI for tests or exams you would not be expected to reduce your solution using all the fancy trig identities - but you *would* be expected to use the addition identities to compute the transform in the first place.

Note: There's a more straightforward way of obtaining the inverse transform. We can use convolution integrals.