

Computing Eigenvalues and Eigenvectors

Example 1:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}.$$

Solution:

The eigenvalues λ and eigenvectors \vec{v} satisfy the matrix equation $A\vec{v} = \lambda\vec{v}$ or, re-arranging terms,

$$(A - \lambda I)\vec{v} = 0, \text{ where } I \text{ is the } 2 \times 2 \text{ identity matrix } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order for $(A - \lambda I)\vec{v} = 0$ to have a non-trivial (i.e., non-zero) solution, a necessary and sufficient condition is:

$$\det(A - \lambda I) = 0.$$

For an $n \times n$ matrix A , this will give us an n^{th} -order polynomial, the characteristic equation, whose roots are the eigenvalues λ . In this case $n = 2$. Finding this characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \left\{ \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} &= 0 \\ \det \begin{pmatrix} 2 - \lambda & -3 \\ 1 & -2 - \lambda \end{pmatrix} &= 0 \\ (2 - \lambda)(-2 - \lambda) + 3 &= 0 \\ \lambda^2 - 1 &= 0. \end{aligned}$$

Therefore our characteristic equation is $\lambda^2 - 1 = 0$. Its roots, $\lambda = \pm 1$, are the eigenvalues. Now we need to find corresponding eigenvectors.

To find the eigenvector \vec{v}_1 corresponding to the eigenvalue $\lambda_1 = 1$ we must solve $(A - \lambda_1 I)\vec{v}_1 = 0$. Substituting for A and λ_1 gives

$$\begin{aligned} \left[\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Note that this matrix equation is equivalent to the single scalar equation $u_1 - 3u_2 = 0$. Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = s$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = 3s$. Thus the eigenvectors associated with the eigenvalue $\lambda_1 = 1$ can be written as

$$\vec{v}_1 = s \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

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Aside: Notice that eigenvectors are unique but only up to a multiplicative factor!

Showing this: Assume \vec{v} is the eigenvector associated with the eigenvalue λ of the matrix A . We can show that $s\vec{v}$ is also an eigenvector associated with the eigenvalue λ of the matrix A for any scalar s : We know that λ, \vec{v} satisfy the equation $A\vec{v} = \lambda\vec{v}$ (by the definition of eigenvalues of eigenvectors/eigenvalues). Multiplying both sides by the scalar s we obtain $sA\vec{v} = s\lambda\vec{v}$. Exploiting commutativity of scalar multiplication, we can write this as $A(s\vec{v}) = \lambda(s\vec{v})$. Thus $s\vec{v}$ is also an eigenvector associated with the eigenvalue λ of the matrix A .

Note: ALL eigenvectors associated with the eigenvalue $\lambda_1 = 1$, when joined with the zero vector, form a subspace. The same is true for $\lambda_2 = -1$. These subspaces are called *eigenspaces*.

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To find the eigenvector \vec{v}_2 corresponding to the eigenvalue $\lambda_2 = -1$ we must solve $(A - \lambda_2 I)\vec{v}_2 = 0$. Substituting for A and λ_2 gives

$$\left[\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Note that this matrix equation is equivalent to the single scalar equation $u_1 - u_2 = 0$. Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = s$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = s$. Thus the eigenvectors associated with the eigenvalue $\lambda_2 = -1$ can be written as

$$\vec{v}_1 = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Answer: The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors for $\lambda_1 = 1$ are $\vec{v}_1 = s \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and the corresponding eigenvectors for $\lambda_2 = -1$ are $\vec{v}_2 = t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, for arbitrary scalar parameters s and t .

Example 2:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}.$$

Solution:

The eigenvalues λ and eigenvectors \vec{v} satisfy the matrix equation $A\vec{v} = \lambda\vec{v}$ or, re-arranging terms,

$$(A - \lambda I)\vec{v} = 0, \text{ where } I \text{ is the } 3 \times 3 \text{ identity matrix } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order for $(A - \lambda I)\vec{v} = 0$ to have a non-trivial (i.e., non-zero) solution, a necessary and sufficient condition is:

$$\det(A - \lambda I) = 0.$$

For an $n \times n$ matrix A , this will give us an n^{th} -order polynomial, the characteristic equation, whose roots are the eigenvalues λ . In this case $n = 2$. Finding this characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \left\{ \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} &= 0 \\ \det \begin{pmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)[- \lambda(5-\lambda) + 4] - 2[(5-\lambda) - 4] - 1[-4 + 4\lambda] &= 0 \\ (\lambda - 1)(\lambda - 2)(\lambda - 3) &= 0, \end{aligned}$$

after some simplification. The roots of this characteristic equation correspond to the eigenvalues of A . Thus we find that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

Now we need to find the eigenvectors. To find the eigenvector \vec{v}_1 corresponding to the eigenvalue $\lambda_1 = 1$ we must solve $(A - \lambda_1 I)\vec{v}_1 = 0$. Substituting for A and λ_1 gives

$$\begin{aligned} \left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\vec{v}_1 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $2u_2 - u_3 = 0$ and (2) $u_1 - u_2 + u_3 = 0$. We can obtain solutions by setting an arbitrary value to one of the u_1 , u_2 , or u_3 - say $u_2 = s$. Then from (1) we find $u_3 = 2s$; then after that from (2) we find $u_1 = -s$. Thus the eigenvectors associated with the eigenvalue $\lambda_1 = 1$ can be written as

$$\vec{v}_1 = s \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ (see note in previous problem concerning uniqueness of eigenvectors only up to multiplicative constants).

To find the eigenvector \vec{v}_2 corresponding to the eigenvalue $\lambda_2 = 2$ we must solve $(A - \lambda_2 I)\vec{v}_2 = 0$. Substituting for A and λ_2 gives

$$\left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\vec{v}_2 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $u_1 - 2u_2 + u_3 = 0$ and (2) $4u_1 - 4u_2 + 3u_3 = 0$. We can obtain solutions by setting an arbitrary value to one of the u_1 , u_2 , or u_3 - say $u_2 = s$. Then from solving (1) we find $u_3 = 2s - u_1$; plugging that into (2) we find $4u_1 - 4s + 3(2s - u_1) = 0$ or $u_1 = -2s$. Then $u_3 = 4s$. Thus the eigenvectors associated with the eigenvalue $\lambda_2 = 2$ can be written as

$$\vec{v}_2 = s \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

Finally, to find the eigenvector \vec{v}_3 corresponding to the eigenvalue $\lambda_3 = 3$ we must solve $(A - \lambda_3 I)\vec{v}_3 = 0$. Substituting for A and λ_3 gives

$$\left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\vec{v}_3 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $-2u_1 + 2u_2 - u_3 = 0$ and (2) $u_1 - 3u_2 + u_3 = 0$. We can obtain solutions by setting an arbitrary value to one of the u_1 , u_2 , or u_3 - say $u_1 = s$. Then from solving (1) we find $u_3 = -2s + 2u_2$; plugging that into (2) we find $s - 3u_2 + (-2s + 2u_2) = 0$ or $u_2 = -s$. Then $u_3 = -4s$. Thus the eigenvectors associated with the eigenvalue $\lambda_3 = 2$ can be written as

$$\vec{v}_3 = s \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$.

Answer: The matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The corresponding eigenvectors for $\lambda_1 = 1$ are $\vec{v}_1 = r \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, the corresponding eigenvectors for $\lambda_2 = 2$ are $\vec{v}_2 = s \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, and the corresponding eigenvectors for $\lambda_3 = 2$ are $\vec{v}_3 = t \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$, for arbitrary scalar parameters r , s and t .