# Inverse Laplace Transform

Below we sketch the steps required to compute the inverse Laplace transform directly. Note that this is very much a summary and is neither thorough nor rigorous. For more careful explanations (and details on the theorems used) please see Churchill and Brown's *Complex Variables and Applications* or Churchill's *Operational Mathematics* (or pretty much any complex analysis textbook).

In order to invert the Laplace transform using complex variables, we first need a representation of $F(s) = \mathscr{L}\{f(t)\}$. Suppose $F(s)$ is the Laplace transform of the piecewise continuous function $f(t)$ of exponential order, that is analytic on and the to right of the line $\mathbb{R}(z) = a$ (see Figure 1). Then by Cauchy's integral formula,

$$\begin{aligned} F(s) &= \frac{1}{2\pi i}\oint_C \frac{F(z)}{z-s}\,dz \\ &= \frac{1}{2\pi i}\int_{C_\alpha} \frac{F(z)}{z-s}\,dz + \frac{1}{2\pi i}\int_{a+ib}^{a-ib} \frac{F(z)}{z-s}\,dz. \end{aligned}$$

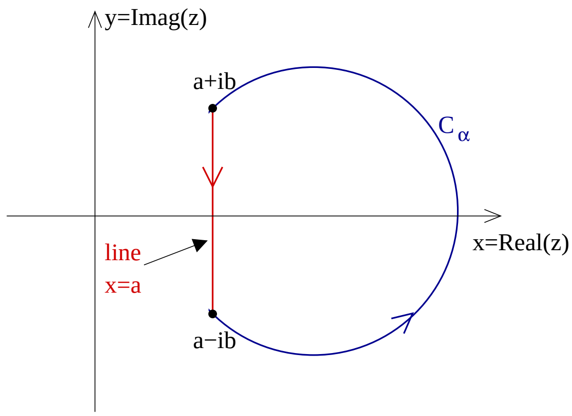


Figure 1: Integration curves for use with Cauchy's integral formula.

Now, since $F(z)$ is analytic to the right of $\mathbb{R}(z) = a$, it is analytic and therefore continuous on $C_\alpha$. This means that $F(z)$ is bounded on $C_\alpha$, $|F(z)| \leq M$ on $C_\alpha$ for some constant $M$. Then by the ML (maximum-length) theorem,

$$|\frac{1}{2\pi i}\int_{C_\alpha} \frac{F(z)}{z-s}\,dz| \leq \frac{M\pi b}{\min(|z-s|)}$$

But $|z-s| = |z-a-(s-a)| \geq |z-a| - |s-a| \geq b - |s-a|$. So

$$\begin{aligned} |\frac{1}{2\pi i}\int_{C_\alpha} \frac{F(z)}{z-s}\,dz| &\leq \frac{M\pi b}{b-|s-a|} \\ &\leq \frac{M\pi}{1-|s-a|/b} \\ &\to 0 \text{ as } b \to \infty. \end{aligned}$$

The last step uses $M \to 0$ as $b \to \infty$ which we can say since $F(z)$ is the Laplace transform of the piecewise continuous $f(t)$ of exponential order. Thus,

$$F(s) = \frac{1}{2\pi i}\int_{a+ib}^{a-ib} \frac{F(z)}{z-s}\,dz \text{ as } b \to \infty$$

We can re-write this as

$$F(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{F(z)}{s-z}\, dz.$$

Now we can invert the transform to recover $f(t)$, $f(t) = \mathscr{L}^{-1}\{F(s)\}$:

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\{F(s)\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{F(z)}{s-z}\, dz\right\} \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(z)\, \mathscr{L}^{-1}\left\{\frac{1}{s-z}\right\} dz \end{aligned}$$

But we know that $\mathscr{L}^{-1}\{1/(s-z)\} = e^{zt}$. Thus,

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(z) e^{tz}\, dz$$

is the line integral that gives our inverse Laplace transform. This is a Bromwich integral and is sometimes called Mellin's inverse formula.

To compute the complex integral, we use the residue theorem. Let $C_\beta$ be a unit circle centred at the origin (see Figure 2).

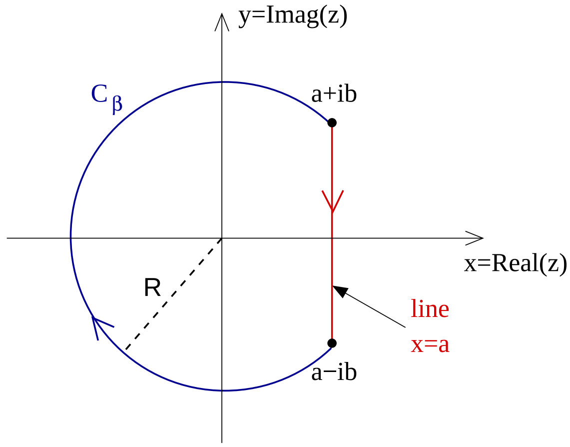


Figure 2: Integration curves for use in computing the inverse transform.

$$\begin{aligned} \int_{a-ib}^{a+ib} F(z) e^{tz}\, dz &= \oint_C F(z) e^{zt}\, dz - \int_{C_\beta} F(z) e^{zt}\, dz \\ &= 2\pi i \sum_{j=1} \kappa_j - \int_{C_\beta} F(z) e^{zt}\, dz, \end{aligned}$$

where the $\kappa_j$ are the residues of $F(z)e^{zt}$ at the singularities of $F(z)$. Now, by Jordan's lemma, if $|F(z)| \leq M/R^k$ for some $k > 0$ on $C_\beta$ (show every time you perform inversion), then

$$\lim_{R\to\infty} \int_{C_\beta} F(z) e^{tz}\, dz = 0.$$

Thus,

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(z)e^{tz}\,dz = \sum_{j=1} \kappa_j.$$

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Key bits here are Cauchy's theorem, Cauchy's integral formula, and the residue theorem. We will state them without proof. When using the residue theorem: there are a few ways to calculate residues depending on the type of singularities of $F(z)$ - see the textbooks mentioned above (or pretty much any other complex variables textbook).

**Cauchy's Theorem or Cauchy-Goursat Theorem:**

If $f(z)$ is analytic within and on a simple (non-intersecting) closed curve $C$ s.t. $f'(z)$ is also continuous there, then

$$\oint_C f(z)\,dz = 0.$$

For multiply connected domains (MCD) (see Figure 3), we instead have

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0.$$

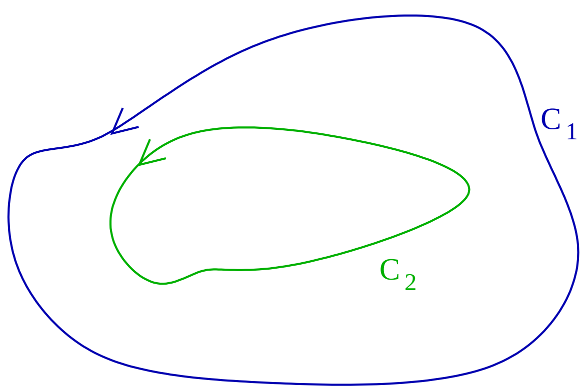


Figure 3: Multiply connected domain for Cauchy-Goursat Theorem.

**Cauchy's Integral Formula:**

If $f(z)$ is analytic within and on a simple closed curve $C$, then for $z_0$ any interior point in $C$,

$$\left.\frac{d^n f}{dz^n}\right|_{z=z_0} = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}.$$

Note that this is INCREDIBLY useful. It means that you can evaluate derivatives by integrating a line integral in complex space. When performing numerical computations, numerical differentiation is very unwieldy. On the other hand, numerical integration is stable and fairly simple to implement.

**Residue Theorem:**

Assume $f(z)$ is analytic within and on a simple closed curve $C$ except for at isolated singularities $z_1$, $z_2$,$z_3$,...,$z_n$within $C$. Draw non-intersecting circles $C_1$, $C_2$, ..., $C_n$ all lying within $C$. We now have a multiply connected domain (MCD) such that $f(z)$ is analytic inside it and on its boundary (see Figure 4).

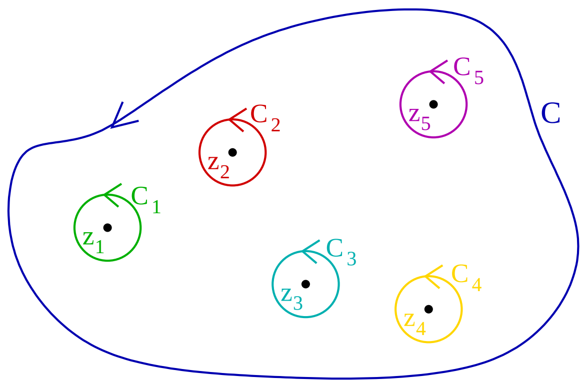


Figure 4: Example of domain for residue theorem with n=5 isolated singularity points (at $z_1$, $z_2$, $z_3$, $z_4$, and $z_5$).

Then by the Cauchy-Goursat theorem,

$$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \ldots - \oint_{C_n} f(z)\,dz = 0$$

$$\text{OR} \oint_C f(z)\,dz - \sum_{j=1}^{n} \oint_{C_j} f(z)\,dz = 0.$$

Since $z_1$, $z_2$,$z_3$,...,$z_n$ are isolated singularities we have Laurent series and can compute the residues. Define

$$\kappa_j = \frac{1}{2\pi i} \oint_{C_j} f(z)\,dz.$$

$\kappa_j$ are the residues of $f(z)$ at $z_j$. Thus

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \kappa_j.$$

This result is valid as $n \to \infty$ for infinite (or semi-infinite) domains, provided the Laurent series converge.