PROBLEM SESSION 1

Problem 1:

Solve the initial value problem $y' + 2y = 0$ with $y(0) = 1$.

Solution:

Using integrating factor $\mu(t) = e^{2t}$, or by inspection, the solution is found to be $y = Ce^{-2t}$. Using the initial condition, the solution is $y = e^{-2t}$.

OR

Use separation of variables:

$$\frac{dy}{dt} = -2y \Rightarrow \frac{dy}{y} = -2dt$$

integrating both sides, find $\ln(y) = -2t + C$. Taking $e^{(\cdot)}$ both sides, $y(t) = \tilde{C}e^{-2t}$. Using the initial condition, the solution is $y = e^{-2t}$.

Problem 2:

Find the general solution of $t^3y' + 4t^2y = e^t$.

Solution:

In standard form, this is $y' + (4/t)y = e^t/t^3$. The integrating factor is found to be

$$\mu(t) = exp\left\{\int \frac{4}{t}\,dt\right\} = t^4.$$

Multiplying by the integrating factor, we can re-write our equation as $t^4y' + 4t^3y = te^t \Rightarrow$

$$\frac{d}{dt}\left(t^4y\right) = te^t.$$

Integrating both sides and simplifying, we find $y(t) = (t-1)e^t/t^4 + Ct^4$ for an arbitrary constant C. This is the general solution to the given ODE.

Problem 3:

Solve the initial value problem $(3x^2 - y^2)dx + (xy - x^3/y)dy = 0$ with $y(1) = 1$.

Solution:

First we solve for dy/dx and simplify:

$$\frac{dy}{dx} = \frac{3x^2y - y^3}{x^3 - xy^2}.$$

Check if the equation is homogeneous, i.e. the RHS can be written as a function of y/x only. Multiply top and bottom by $1/x^n$ where n is the highest degree in y (here $n = 3$, multiply top and bottom by $1/x^3$) to obtain

$$\frac{dy}{dx} = \frac{3(y/x) - (y/x)^3}{1 - (y/x)^2}.$$

The equation is homogeneous. Pose the substitution $v = y/x$. Then $dy/dx = v + xdv/dx$ and the ode becomes

$$v + x\frac{dv}{dx} = \frac{3v - v^3}{1 - v^2}.$$

Solve using separation of variables to obtain $\ln(v)/2 - v^2/4 = \ln(x) + C$. We can't solve for v! Putting back in terms of y ($v = y/x$), we have $\ln(y)/2 - y^2/4x^2 = 3\ln(x)/2 + C$. Using the IC $y(1) = 1$ we find $C = -1/4$. Thus the solution to the IVP is

$$\frac{y^2}{4x^2} - \frac{1}{2}\ln(y) = \frac{1}{4} - \frac{3}{2}\ln(x).$$

We could simplify further but we can't solve for y - leave solution in implicit form.

Problem 4:

In a murder investigation a corpse was found by Inspector Clouseau at exactly 8:00 p. m. Being alert, he measures the temperature of the body and finds it to be 70F (Fahrenheit). Two hours later, Inspector Clouseau again measures the temperature of the corpse and finds it to be 60F. If the room temperature is 50 F, and we assume that Newton's law of cooling applies, when did the murder occur? (Assume that the temperature of the body at the time of the murder was 98.6F).

Solution:

Let $T(t)$ be the temperature of the body in degrees Fahrenheit starting at $t = 0$, which measures hours starting from 8:00p.m. Then, $T(t)$ satisfies

$$T' = -k(T-50), \;\; T(0) = 70.$$

We also know that $T(2) = 60$. The solution to the IVP above is

$$T = 50 + 20e^{-kt}.$$

Now set $T = 60$, when $t = 2$ to determine k as

$$k = \ln(2)/2 \approx 0.34657...$$

Now we want to find t such that $T = 98.6$. Clearly, this value of t will satisfy t < 0, as the murder occurs before 8:00p.m. Thus,

$$98.6 = 50 + 20e^{-kt}, \;\; t = -\ln(48.6/20)/k = -2.562 \approx -2.6.$$

Thus, the murder occurs approximately 156 minutes before 8:00p.m. or at 5:24p.m.

Problem 5:

A radioactive substance emits radiation and changes to a new substance. The amount of ra- dioactivity is proportional to the quantity of the original substance remaining. The half-life of a radioactive substance is the amount of time it takes for half the substance to decay radioactively. If the half-life of radium is 1760 years, how much radium is left from an initial gram of radium after 100 years?

Solution:

The amount of radioactivity $R = R(t)$, in grams, satisfies $R' = -kR$, $R(0) = R_0$, where $k > 0$ is constant, t is time measured in years, and R_0 is the initial amount of substance. For us, $R(0) = 1$ gram. The solution is

$$R(t) = R_0 e^{-kt}.$$

We are given that $R = R_0/2$, when $t = 1760$ years. This allows us to find k from the above as

$$e^{-1760k} = ln(1/2) \;\; \rightarrow \;\; k = ln(2)/1760 \approx .000394.$$

The units of k are years^{-1}. Now find R when $t = 100$ years. We get from the equation above with $R_0 = 1$ that

$$R = exp\left(-\ln(2100/1760)\right) \approx .96\,\text{grams}.$$

Hence, very little of the original material has dissipated after 100 years.