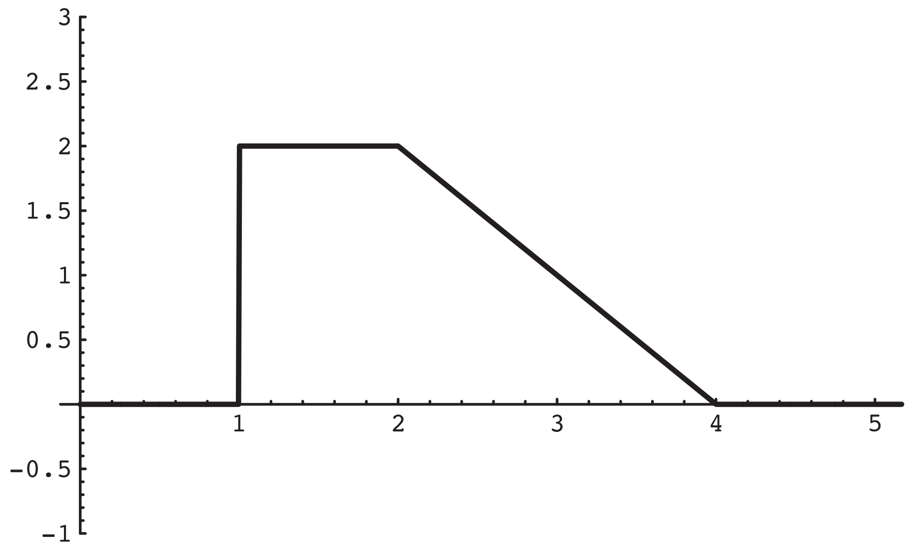
Problem Session 3

Problem 1:Find the Laplace transform of the function shown in the graph:



Solution:

What we'll do: write this function in terms of inequalities, then in terms of step functions, THEN take the Laplace transform.

First, writing down this equation in terms of inequalities:

$$y(t) = \begin{cases} 0, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 4-t, & 2 \le t < 4 \\ 0. & t \ge 4. \end{cases}$$

Re-writing in terms of step functions we obtain:

$$y(t) = 2u_1(t) + (2-t)u_2(t) + (t-4)u_4(t).$$

Now we compute the Laplace transform. Let $\mathscr{L}\{y(t)\} = Y(s)$. Then

$$\begin{aligned} \mathscr{L}\{y(t)\} &= \mathscr{L}\{2u_1(t) + (2-t)u_2(t) + (t-4)u_4(t)\} \\ Y(s) &= 2\mathscr{L}\{u_1(t)\} - \mathscr{L}\{(t-2)u_2(t)\} + \mathscr{L}\{(t-4)u_4(t)\}, \end{aligned}$$

exploiting the linearity of the Laplace transform. Now, recall that $\mathscr{L}\{u_c(t)\} = e^{-cs}/s$ and $\mathscr{L}\{f(t-c)u_c(t)\} = e^{-cs}F(s)$, where $F(s) = \mathscr{L}\{f(t)\}$ (or $\mathscr{L}\{g(t)u_c(t)\} = e^{-cs}\mathscr{L}\{g(t+c)\}$ Then going term-by-term:

$$\mathscr{L}\{u_1(t)\} = e^{-s}/s$$
$$\mathscr{L}\{(t-2)u_2(t)\} = e^{-2s}\mathscr{L}\{t\} = e^{-2s}/s^2$$
$$\mathscr{L}\{(t-4)u_4(t)\} = e^{-4s}\mathscr{L}\{t\} = e^{-4s}/s^2.$$

Therefore the Laplace transform of the function $y(t)$ shown in the graph is

$$Y(s) = \frac{2se^{-s} - e^{-2s} + e^{-4s}}{s^2}.$$

Problem 2: Solve the following initial value problems using the method of Laplace transforms:

$$y'' - y' + 2y = -8\cos(t) - 2\sin(t),\ y(\pi/2) = 1,\ y'(\pi/2) = 0.$$

Solution: Let $w(t) = y(t+\pi/2)$. Then: $w'' - w' + 2w = 8\sin(t) - 2\cos(t)$, $w(0) = 1$, $w'(0) = 0$, noting that $\cos(t + \pi/2) = -\sin(t)$ and $\sin(t+\pi/2) = \cos(t)$. Now that the initial data is at $t = 0$, we can use the method of Laplace transforms.

Let $W(s) = \mathscr{L}\{w(t)\} = \int_0^\infty w(t)e^{-st}\,dt$.
We have that $\mathscr{L}\{w''\} = s^2W(s) - sw(0) - w'(0) = s^2W(s) - s$, $\mathscr{L}\{w'\} = sW(s) - w(0) = sW(s) - 1$, $\mathscr{L}\{\sin(t)\} = 1/(s^2+1)$, and $\mathscr{L}\{\cos(t)\} = s/(s^2+1)$. Then

$$\mathscr{L}\left\{w'' - w' + 2w\right\} = \mathscr{L}\{8\sin(t) - 2\cos(t)\} \ \Rightarrow\ \mathscr{L}\left\{w''\right\} - \mathscr{L}\left\{w'\right\} + 2\mathscr{L}\{w\} = 8\mathscr{L}\{\sin(t)\} - 2\mathscr{L}\{\cos(t)\}$$

$$\Rightarrow (s^2W(s) - s) - (sW(s) - 1) + 2W(s) = 8\left(\frac{1}{s^2+1}\right) - 2\left(\frac{s}{s^2+1}\right).$$

Solving for $W(s)$:

$$(s^2 - s + 2)W(s) = \left(\frac{8}{s^2+1}\right) - \left(\frac{2s}{s^2+1}\right) + s - 1 \ \Rightarrow\ W(s) = \frac{s^3 - s^2 - s + 7}{(s^2+1)(s^2-s+2)}.$$

Before inverting split $W(s)$ up into "manageable bits" by using a partial fractions expansion. Note that $s^2 - s + 2 = (s-1/2)^2 + 7/4$ which is the form of the denominator for the laplace transform of $e^{t/2}\sin(\sqrt{7}t/2)$ and $e^{t/2}\cos(\sqrt{7}t/2)$.

$$\frac{s^3 - s^2 - s + 7}{(s^2+1)(s^2-s+2)} = \frac{As+B}{s^2+1} + \frac{C(s-1/2) + (\sqrt{(7)}/2)D}{(s-1/2)^2 + 7/4} \ \Rightarrow\ A = 3,\ B = 5,\ C = -2,\ D = -\frac{8}{\sqrt{7}}.$$

Now we invert:

$$w(t) = \mathscr{L}^{-1}\{W(s)\} = \mathscr{L}^{-1}\left\{\frac{s^3 - s^2 - s + 7}{(s^2+1)(s^2-s+2)}\right\} \tag{1}$$

$$= 3\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 5\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s-1/2}{(s-1/2)^2+7/4}\right\} - \frac{8}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}/2}{(s-1/2)^2+7/4}\right\}. \tag{2}$$

Noting from tables that

$$\mathscr{L}^{-1}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2} \ \text{ and } \ \mathscr{L}^{-1}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2},$$

we can write down $w(t)$:

$$w(t) = 3\cos(t) + 5\sin(t) - 2e^{t/2}\cos(\sqrt{7}t/2) - (8/\sqrt{7})e^{t/2}\sin(\sqrt{7}t/2).$$

But we want $y(t)$. Well, since $w(t) = y(t+\pi/2)$, $y(t) = w(t-\pi/2)$ and thus:

$$y(t) = 3\cos(t-\pi/2) + 5\sin(t-\pi/2) - 2e^{(t-\pi/2)/2}\cos(\sqrt{7}(t-\pi/2)/2) - (8/\sqrt{7})e^{(t-\pi/2)/2}\sin(\sqrt{7}(t-\pi/2)/2),$$

or, noting that $\cos(t-\pi/2) = \sin(t)$, $\sin(t-\pi/2) = -\cos(t)$, $\cos(\sqrt{7}(t-\pi/2)/2) = \cos(\sqrt{7}t/2)\cos(\sqrt{7}\pi/4) + \sin(\sqrt{7}t/2)\sin(\sqrt{7}\pi/4)$, and $\sin(\sqrt{7}(t-\pi/2)/2) = \sin(\sqrt{7}t/2)\cos(\sqrt{7}\pi/4) - \cos(\sqrt{7}t/2)\sin(\sqrt{7}\pi/4)$,

$$y(t) = 3\sin(t) - 5\cos(t) - e^{(\frac{t}{2}-\frac{\pi}{4})}\left\{\left(2\cos(\frac{\sqrt{7}\pi}{4}) - \frac{8}{\sqrt{7}}\sin(\frac{\sqrt{7}\pi}{4})\right)\cos(\frac{\sqrt{7}t}{2}) + \left(2\sin(\frac{\sqrt{7}\pi}{4}) + \frac{8}{\sqrt{7}}\cos(\frac{\sqrt{7}\pi}{4})\right)\sin(\frac{\sqrt{7}t}{2})\right\}$$

Problem 3: **Control Theory Problem**

Consider a servomechanism that models an automatic pilot. Such a mechanism applies torque to the steering control shaft so that a plane or boat will follow the prescribed course. If we let $y(t)$ be the true direction (i.e. the angle) of the craft at time t and $g(t)$ be the desired direction at time t, then

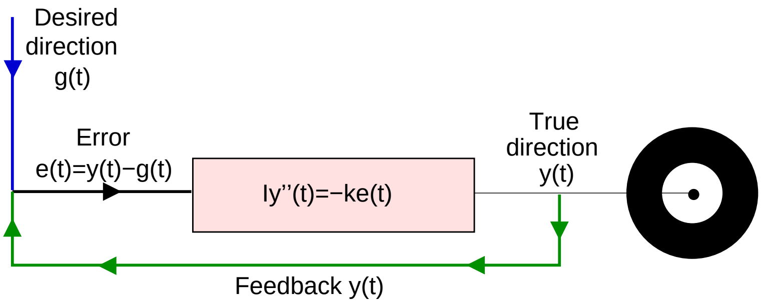
$$e(t) = y(t) - g(t)$$

denotes the **error** or **deviation** between the desired direction and the true direction.

Assume that the servomechanism can measure the error $e(t)$ and feed back to the steering shaft a component of torque that is proportional to $e(t)$, but opposite in sign (see figure). The rotational analogue of Newton's second law states that: (moment of inertia I)$\times$(angular acceleration)=(total torque). For our servomechanism this becomes:

$$Iy''(t) = -ke(t),$$

where k is a positive proportionality constant.



Using the method of Laplace transforms, determine the error $e(t)$ for the automatic pilot if the steering shaft is initially at rest in the zero direction ($y(0) = 0, y'(0) = 0$) and the desired direction is given by $g(t) = at$ for a a constant.

Solution:

Note that $e(t) = y(t) - g(t)$ where $g(t) = at$ but $y(t)$ remains unknown. To find $e(t)$ we must first solve for $y(t)$. Take the Laplace transform of both sides of the differential equation:

$$\mathscr{L}\{Iy''\} = \mathscr{L}\{-ke(t)\} \Rightarrow I\mathscr{L}\{y''\} = -k\mathscr{L}\{y(t) - g(t)\} \Rightarrow I\mathscr{L}\{y''\} = -k(\mathscr{L}\{y(t)\} - \mathscr{L}\{g(t)\}).$$

Letting $Y(s) = \mathscr{L}\{y(t)\} = \int_0^\infty y(t)e^{-st}\,dt$,, the Laplace transforms are:
$\mathscr{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$ as $y(0) = y'(0) = 0$
$\mathscr{L}\{g(t)\} = \mathscr{L}\{at\} = a\mathscr{L}\{t\} = a/s^2$ as $g(t) = at$
Plugging back into the differential equation,

$$Is^2Y(s) = -kY(s) + \frac{ak}{s^2}.$$

Solving for $Y(s)$,

$$(Is^2 + k)Y(s) = \frac{ak}{s^2} \Rightarrow Y(s) = \frac{ak}{s^2(Is^2 + k)}.$$

Now we must invert to find $y(t)$, that is, $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. Re-write $Y(s)$ so that it is in a form we can recognize from tables, using a partial fractions expansion:

$$\frac{ak/I}{s^2(s^2 + k/I)} = \frac{A}{s^2} + \frac{B\sqrt{k/I}}{s^2 + k/I}.$$

Obtain $A = a$ and $B = -a/\sqrt{k/I}$. Here we used that $s^2 + k/I$ is the form of the denominator for the transform of $\sin(\sqrt{k/I}t)$. So

$$Y(s) = \frac{a}{s^2} - \left(a\sqrt{\frac{I}{k}}\right)\frac{\sqrt{k/I}}{s^2 + (k/I)}.$$

Thus,

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{a}{s^2} - \left(a\sqrt{\frac{I}{k}}\right)\frac{\sqrt{k/I}}{s^2+(k/I)}\right\} = a\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} - a\sqrt{\frac{I}{k}}\mathscr{L}^{-1}\left\{\frac{\sqrt{k/I}}{s^2+(k/I)}\right\}$$

$$\Rightarrow\ y(t) = at - a\sqrt{\frac{I}{k}}\sin(\sqrt{k/I}t).$$

Finally our goal is to determine the error $e(t)$ for the automatic pilot, $e(t) = y(t) - g(t)$ where $g(t) = at$:

$$e(t) = -a\sqrt{\frac{k}{I}}\sin\left(\sqrt{\frac{k}{I}}t\right).$$

Problem 4:
Use the convolution theorem to find $\mathscr{L}^{-1}\left\{1/(s^2+1)^2\right\}$.
Solution:

Write

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right).$$

Since $\mathscr{L}^{-1}\left\{1/(s^2+1)\right\} = \sin(t)$, it follows from the convolution theorem that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \sin(t)\star\sin(t) \\ &= \int_0^t \sin(t-\tau)\sin(\tau)\,d\tau\end{aligned}$$

Now, $sin(t-\tau)\sin(\tau) = \cos(t-2\tau)/2 - \cos(t)/2$ using the trig ID $\sin A \sin B = \cos(A-B) - \cos(A+B))/2$, so the integral becomes:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \frac{1}{2}\int_0^t (\cos(t-2\tau) - \cos(t))\,d\tau \\ &= \frac{1}{2}\left[\frac{\sin(t-2\tau)}{-2}\right]_0^t - \frac{t\cos(t)}{2} \\ &= \frac{\sin(t) - t\cos(t)}{2}\end{aligned}$$

Thus the $\mathscr{L}^{-1}\left\{1/(s^2+1)^2\right\} = (\sin(t) - t\cos(t))/2$.

Problem 5:
Solve the integro-differential equations

$$y'(t) = 1 - \int_0^t y(t-v)e^{-2v}dv. \; y(0) = 1.$$

Solution:
Notice that the integral $\int_0^t y(t-v)e^{-2v}dv$ is the convolution of $y(t)$ and e^{-2v}. Re-write (for convenience) as $y(t) \star e^{-2t}$. Now we solve using Laplace transforms. Let $Y(s) = \mathscr{L}\{y(t)\}$. Taking the Laplace transform of our equation:

$$\begin{aligned} \mathscr{L}\{y'(t)\} &= \mathscr{L}\{1 - y(t) \star e^{-2t}\} \\ &= \mathscr{L}\{1\} - \mathscr{L}\{y(t) \star e^{-2t}\} \end{aligned}$$

Computing the transforms:

$$\mathscr{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1,$$

$$\mathscr{L}\{1\} = \frac{1}{s},$$

$$\text{and } \mathscr{L}\{y(t) \star e^{-2t}\} = Y(s)\left(\frac{1}{s+2}\right).$$

The last transform we obtain using the convolution theorem: $\mathscr{L}\{f \star g\} = F(s)G(s) = \mathscr{L}\{f\}\mathscr{L}\{g\}$. And we know $\mathscr{L}\{e^{-2t}\} = 1/(s+2)$.
Thus our transformed IVP is:

$$sY(s) - 1 = \frac{1}{s} - Y(s)\left(\frac{1}{s+2}\right).$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1},$$

using partial fraction expansions. We find our solution taking the inverse transform...

$$y(t) = 2 - e^{-t}.$$