REDUCTION OF ORDER EXAMPLE

Problem:
Given that $y_1(x) = 1/x$ is a solution to $x^2y'' - 2xy' - 4y = 0,\ x > 0$, determine a second linearly independent solution.
Solution:
Using reduction of order, let $y_2(x) = v(x)y_1(x) = v(x)/x$. Our goal is to find $v(x)$ so that $y_2(x)$ is the second linearly independent solution.
To this end we plug $y_2(x)$ into the ODE in standard form. The ODE in standard form is

$$y'' - \frac{2}{x}y' - \frac{4}{x^2}y = 0$$

. Since $y_2' = -v/x^2 + v'/x$ and $y_2'' = 2v/x^3 - 2v'/x^2 + v''/x$, $x^2y_2'' - 2xy_2' - 4y_2 = 0$ becomes

$$\begin{aligned}(2v/x^3 - 2v'/x^2 + v''/x) - 2(-v/x^2 + v'/x)/x - 4(v/x)/x^2 &= 0 \\ (1/x)v'' + (-4/x^2)v' &= 0,\end{aligned}$$

after some simplification. This is a first order differential equation for v'. To make that clear, let $v' = w$. The ODE becomes

$$\frac{1}{x}w' - \frac{4}{x^2}w = 0.$$

We can solve it using an integrating factor or separation of variables. We'll use the latter method:

$$\begin{aligned}\frac{1}{x}\frac{dw}{dx} - \frac{4}{x^2}w &= 0 \\ \frac{1}{x}\frac{dw}{dx} &= \frac{4}{x^2}w \\ \frac{dw}{w} &= \frac{4}{x}dx.\end{aligned}$$

Integrating both sides,

$$\begin{aligned}\int \frac{dw}{w} &= \int \frac{4}{x}dx \\ \ln|w| &= 4\ln|x| + C\end{aligned}$$

where C is an arbitrary constant. Then taking the exponential of both sides,

$$\begin{aligned}e^{\ln|w|} &= e^{4\ln|x|+C} \\ w &= \tilde{C}x^4.\end{aligned}$$

But recall that $w = v'$ - so we have $v' = \tilde{C}x^4$. Integrating we find $v(x) = \tilde{\tilde{C}}x^5 + D$ where $\tilde{\tilde{C}}$ and D are arbitrary constants. In class we said we were only concerned with the t - dependence and ignored the constants. We were hand-waving - the reason for this is that then $y_2(x) = v(x)y_1(x) = \tilde{\tilde{C}}x^4 + D/x$ and since D/x can be expressed as a linear combination of $y_1(x)$ it is linearly dependent with $y_1(x) = 1/x$ (the Wronskian is zero).
Then $y_2(x) = v(x)y_1(x) = (x^5)(1/x) = x^4$ **is our second linearly independent solution?** Use the Wronskian to check linear independence. $W = y_1y_2' - y_1'y_2 = (1/x)(4x^3) - (-1/x^2)(x^4) = 3x^2 \neq 0$ (as $x > 0$). Plug into the ODE to check if it's a solution $x^2y_2'' - 2xy_2' - 4y_2 = x^2(12x^2) - 2x(4x^3) - 4(x^4) = 0$.
Therefore the general solution is $y(x) = C_1y_1(x) + C_2y_2(x) = C_1/x + C_2x^4$.

Why is linear dependence important? Let's take $v(x) = \tilde{\tilde{C}}x^5 + D$ and see what happens: We found $v(x)$ such that $y_2(x) = v(x)y_1(x)$ satisfies the differential equation. Then $y_2(x) = \tilde{\tilde{C}}x^4 + D/x$ and the general solution becomes

$$\begin{aligned}y(x) &= K_1y_1(x) + K_2y_2(x) \\ &= \frac{K_1}{x} + K_2\left(\tilde{\tilde{C}}x^4 + D/x\right) \\ &= (K_1 + K_2D)\frac{1}{x} + K_2\tilde{\tilde{C}}x^4 \\ &= A_1\frac{1}{x} + A_2x^4\end{aligned}$$

where A_1 and A_2 are arbitrary constants. $\tilde{\tilde{C}}$ and D have been absorbed into the arbitrary constants into the general solution! This is because the D/x part is linearly dependent with $y_1(x)$.