VARIATION OF PARAMETERS EXAMPLE

Problem:

Find the general solution of

$$y'' - 6y' + 9y = x^{-3}e^{3x}$$

Solution:

The general solution for an inhomogeneous equation is split into two parts: $y_p(x)$, the particular solution, and $y_h(x)$, the homogeneous solution (solution of the associated homogeneous equation). After finding both $y_h(x)$ and $y_p(x)$ we recover the general solution,

$$y(x) = y_h(x) + y_p(x).$$

First let's find the homogeneous solution, $y_h(x)$. We do this first b/c: (1) If we're going to use the method of undetermined coefficients to find the particular solution, we want to make sure that our trial expression or "guess" is not a homogeneous solution, and (2) if we use variation of parameters to find the particular solution (and for this example we will), we require the two linearly independent homogeneous solutions. The homogeneous solution $y_h(x)$ satisfies

$$y_h'' - 6y_h + 9y_h = 0.$$

To find the solution, use the substitution $y_h(x) = e^{rx}$. Then $y_h' = re^{rx}$, $y_h'' = r^2e^{rx}$ and our ode becomes $r^2e^{rx} - 6re^{rx} + 9e^{rx} = 0$. Since $e^{rx} \neq 0$, we can divide by e^{rx} and recover the characteristic equation,

$$r^2 - 6r + 9 = 0.$$

Solving for r (either by factoring or by using the quadratic equation) we find that $r = 3$ is a *repeated root.* Therefore the homogeneous solution is

$$y_h(x) = C_1e^{3x} + C_2xe^{3x}$$

and $y_1(x) = e^{3x}$, $y_2(x) = xe^{3x}$ are the two linearly independent solutions to the homogeneous equation.

Next let's find the particular solution y_p. We use variation of parameters. Let

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are the two linearly independent solutions to the homogeneous equation, found above. We could insert our computed y_1 and y_2 at this point if desired. Your choice - we'll do it both ways here. Then $u_1(x)$ and $u_2(x)$ must satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \text{ OR } \begin{pmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ x^{-3}e^{3x} \end{pmatrix}.$$

where $g(x) = x^{-3}e^{3x}$ is our inhomogeneity. Solving for u_1', u_2' (performing a matrix inversion) we find

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W[y_1,y_2]}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ g(x) \end{pmatrix} \text{ OR } \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{e^{6x}}\begin{pmatrix} e^{3x}+3xe^{3x} & -xe^{3x} \\ -3e^{3x} & e^{3x} \end{pmatrix}\begin{pmatrix} 0 \\ x^{-3}e^{3x} \end{pmatrix}$$

where the determinant of the inverted matrix is $y_1y_2' - y_2y_1' = W[y_1,y_2]$, the Wronskian. Note that if correctly computed $y_1(x) = e^{3x}$ and $y_2(x)$, they're linearly independent, and the Wronskian is nonzero. Thus we find

$$u_1' = \frac{-y_2g}{W[y_1,y_2]} \text{ OR } u_1' = \frac{-x^{-2}e^{6x}}{e^{6x}} = -x^{-2}$$

$$u_2' = \frac{y_1g}{W[y_1,y_2]} \text{ OR } u_2' = \frac{x^{-3}e^{6x}}{e^{6x}} = x^{-3}.$$

If we hadn't inserted values yet, we could do so now. $y_1 = e^{3x}$, $y_2 = xe^{3x}$, $g = x^{-3}e^{3x}$, and $W[y_1,y_2] = y_1y_2' - y_2y_1' = (e^{3x})(e^{3x} + 3xe^{3x}) - (xe^{3x})(3e^{3x}) = e^{6x}$. Then,

$$u_1' = -x^{-2}; \text{solving}, u_1 = x^{-1}$$

$$u_2' = x^{-3}; \text{solving}, u_2 = -x^{-2}/2.$$

(We drop constants b/c they would only multiply, and become part of, the homogeneous solution). Thus the particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ is

$$\begin{aligned} y_p(x) &= (x^{-1})(e^{3x}) + (-x^{-2}/2)(xe^{3x}) \\ &= x^{-1}e^{3x} - x^{-1}e^{3x}/2 \\ y_p(x) &= x^{-1}e^{3x}/2. \end{aligned}$$

We can check our particular solution by plugging it back into the ODE.

That's it, we're done! We've found the homogeneous solution $y_h(x)$ and - using variation of parameters - the particular solution $y_p(x)$. The general solution is there sum $y(x) = y_h(x) + y_p(x)$, or

$$y(x) = C_1 e^{3x} + C_2 x e^{3x} + \frac{e^{3x}}{2x}.$$