Math 257/316 Assignment 5
Due Monday October 26th in class

SOLUTIONS

Problem 1: Sketch the odd, even, and full periodic extensions on [3L; 3L] of

(a) e^x, with $L = 1$

(b) $4 - x^2$, with $L = 2$

(c) $g(x) = \begin{cases} 1 + x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$, with $L = 1$.

SOLUTION:

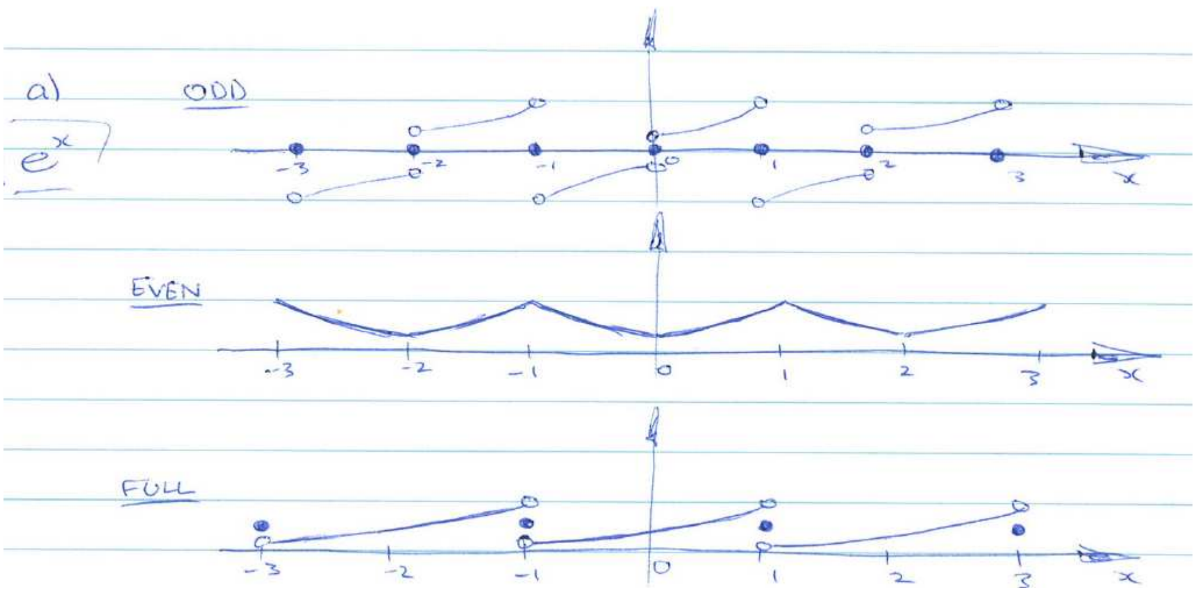

Figure 1: Problem 1a; odd, even, and full periodic extensions of e^x with $L = 1$

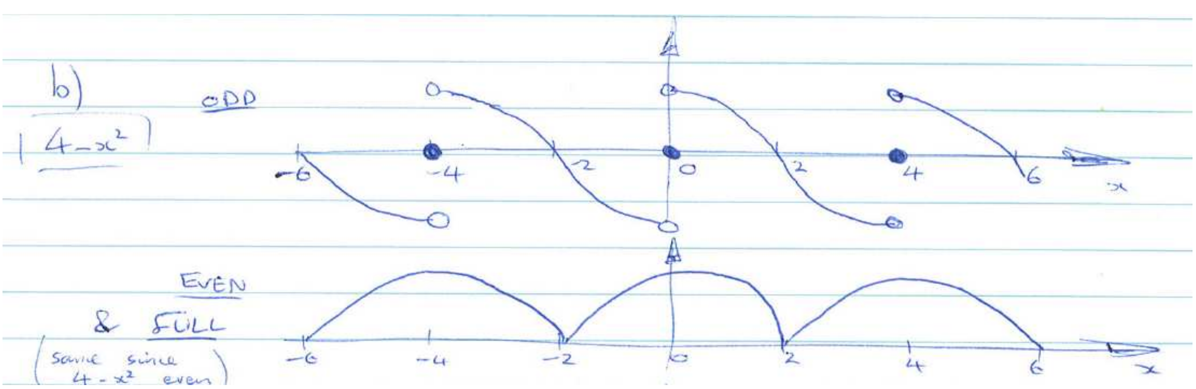

Figure 2: Problem 1b; odd, even, and full periodic extensions of $4 - x^2$ with $L = 2$

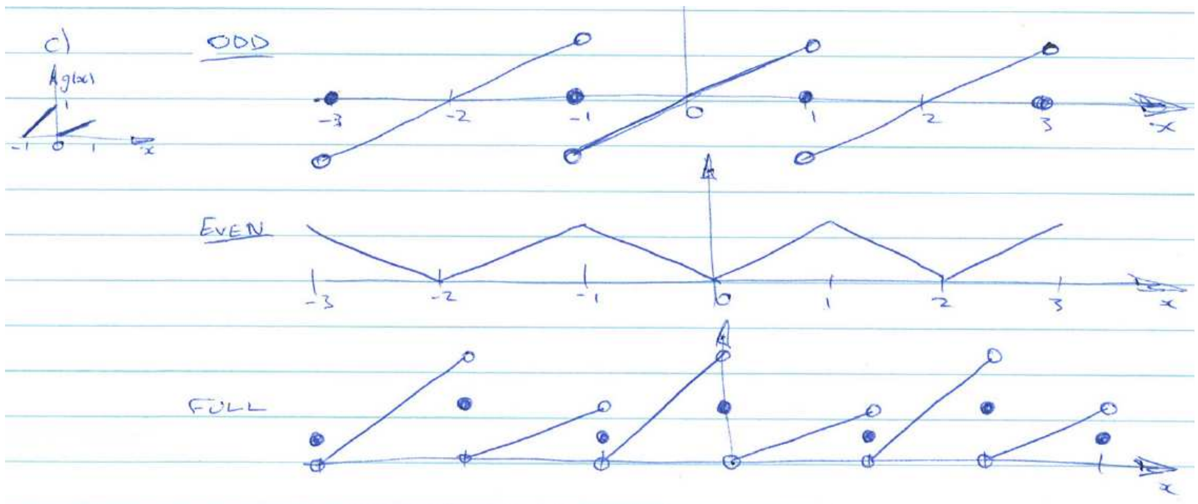

Figure 3: Problem 1a; odd, even, and full periodic extensions of $g(x)$ with $L = 1$

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Problem 2: Chemical diffusion through a thin layer is governed by the equation

$$\frac{\partial C}{\partial t} = k\frac{\partial^2 C}{\partial x^2} - LC$$

where $C(x,t)$ is the concentration in moles/cm^3, the diffusivity k is a positive constant with units cm^2/sec, and $L > 0$ is a consumption rate with units sec^{-1}. Assume boundary conditions are

$$C(0,t) = C(a,t) = 0, \quad t > 0,$$

and the initial concentration is given by

$$C(x,0) = f(x), \quad 0 < x < a.$$

(a) Use the method of separation of variables to solve for the concentration $C(x,t)$.

(b) What happens to the concentration as $t \to \infty$?

(c) What is the concentration $C(x,t)$ if the initial condition is $C(x,0) = \cos(\pi x/a)$?

Hint: It may be useful to know that

$$\int_0^a \sin(n\pi x/a)\cos(\pi x/a)\,dx = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2an}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

SOLUTION:
(a) Use the method of separation of variables to solve for the concentration $C(x,t)$.
We use separation of variables. Let $C(x,t) = X(x)T(t)$. Then $C_t = kC_{xx} - LC$ becomes $X(x)T'(t) = kX''(x)T(t) - LX(x)T(t)$. We divide both sides by $kX(x)T(t)$ and re-arrange to obtain:

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} - \frac{L}{k} = \tilde{\lambda}, \tag{1}$$

$$\text{or } \frac{1}{k}\frac{T'}{T} + \frac{L}{k} = \frac{X''}{X} = \lambda, \tag{2}$$

where $\tilde{\lambda}$, λ are constant. We could use *either* (1 or 2) to get the solution. Since using (2) is more straightforward, that's what we'll use.

What happens to the boundary conditions under the separation of variables?

$$0 = C(0,t) = X(0)T(t) \Rightarrow X(0) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

$$0 = C(a,t) = X(a)T(t) \Rightarrow X(a) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

So we have $X(0) = X(2) = 0$. Can the initial condition tell us anything at this stage?

$$f(x) = C(x,0) = X(x)T(t) \Rightarrow T(t) = f(x)/X(x)???$$

No, it can't. The trick worked on the boundary conditions b/c they were homogeneous ($= 0$). We'll actually use the initial condition at the end to solve for constants.

Let's start with the T-equation from (2):

$$T'(t) = (\lambda k - L)T(t).$$

Solving, we notice that this is a separable equation

$$\frac{dT}{dt} = (\lambda k - L)T \Rightarrow \frac{dT}{T} = (\lambda k - L)dt.$$

Integrating both sides,

$$\int \frac{dT}{T} = \int (\lambda k - L)dt \Rightarrow \ln(T) = (\lambda k - L)t + B \Rightarrow T(t) = Be^{(\lambda k - L)t},$$

taking the exponential of both sides. B is an arbitrary constant.

Next we deal with the X-equation in (2) with conditions $X(0) = X(a) = 0$ derived from the boundary conditions

$$\begin{aligned} X'' &= \lambda X \\ X(0) &= X(a) = 0. \end{aligned}$$

This is an *eigenvalue problem.* There are 3 cases to consider: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

We begin with the $\lambda > 0$: set $\lambda = \mu^2 > 0$. Then $X''(x) - \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 - \mu^2 = 0$, which has roots $r = \pm\mu$.Thus $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \Rightarrow B_1 + B_2 = 0 \\ X(a) &= 0 \Rightarrow B_1 e^{a\mu} + B_2 e^{-a\mu} = 0. \end{aligned}$$

Solving simultaneously we find $B_1 = B_2 = 0$. (The first equation gives $B_2 = -B_1$, plugging into the first equation gives $B_1 e^{2\mu} - B_1 e^{-2\mu} = 0 \Rightarrow B_1(e^{2\mu} - e^{-2\mu}) = 0$, and this means that $B_1 = 0$ because $e^{2\mu} - e^{-2\mu}$is only zero at $\mu = 0$, which it isn't here - $\mu^2 = \lambda > 0$). Thus we have recovered the trivial solution (aka zero solution). Therefore for $\lambda > 0$ we have no eigenvalues or eigenfunctions.

Next we consider the $\lambda = 0$ case (we could consider it jointly with the $\lambda < 0$ or $\lambda > 0$ cases, if we're very careful, but for the purposes of a systematic approach we won't here). Then $X'' = 0 \Rightarrow X(x) = Dx + E$. Applying boundary conditions, $0 = X(0) = E \Rightarrow E = 0$; $0 = X(a) = Da \Rightarrow D = 0$. Thus we have recovered the trivial solution (aka zero solution). Therefore for $\lambda = 0$ we have no eigenvalues or eigenfunctions.

Finally we look at the $\lambda < 0$ case. Set $\lambda = -\mu^2 < 0$. Then $X''(x) + \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 + \mu^2 = 0$, which has roots $r = \pm i\mu$.Thus $X(x) = \tilde{B}_1 e^{i\mu x} + \tilde{B}_2 e^{-i\mu x}$ or $X(x) = B_1 \sin(\mu x) + B_2 \cos(\mu x)$ (for more details on solving this ode, see your textbook, section 3.3). We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \Rightarrow B_1 \sin(0) + B_2 \cos(0) = 0 \Rightarrow B_2 = 0 \\ X(2) &= 0 \Rightarrow B_1 \sin(a\mu) = 0. \end{aligned}$$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \ldots$, the second condition tells us that $a\mu = n\pi$ or $\mu = n\pi/a$, $n = 1, 2, 3, \ldots$ Thus we have our eigenfunctions an eigenvalues for $\lambda < 0$:

$$\begin{aligned} \lambda_n &= -\left(\frac{n\pi}{a}\right)^2 \\ X_n(x) &= \sin(n\pi x/a). \end{aligned}$$

Now we re-assemble. Recall $C(x,t) = X(x)T(t)$. Therefore

$$C_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{a}\right)\exp\left[-\left(\frac{n^2\pi^2}{a^2}k + L\right)t\right]$$

for $n = 1, 2, 3, \ldots$ are each solutions to the pde. The pde is linear so we can use the principle of superposition, and sum them to make up a general solution:

$$C(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)\exp\left[-\left(\frac{n^2\pi^2}{a^2}k + L\right)t\right],$$

where the b_n are constants.

We solve for the b_n using the initial condition. That is, $C(x,0) = f(x)$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right),$$

which is a Fourier sine series. We exploit orthogonality of the sines, that is,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

where $L = a$ to solve for the individual b_n:

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

since $L = a$. And that's it! We don't know $f(x)$ (yet), so we're done. The concentration $C(x,t)$ is

$$C(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)\exp\left[-\left(\frac{n^2\pi^2}{a^2}k + L\right)t\right], \text{ with } b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx.$$

(b) What happens to the concentration as $t \to \infty$?

Well,

$$\lim_{t\to\infty} \exp\left[-\left(\frac{n^2\pi^2}{a^2}k + L\right)t\right] = 0.$$

Therefore, as $t \to \infty$, the concentration $C(x,t) \to 0$. Which makes sense! The equation describes the diffusion of a chemical through a thin layer. Eventually, it all diffuses through, so the concentration goes to zero.

(c) What is the concentration $C(x,t)$ if the initial condition is $C(x,0) = \cos(\pi x/a)$?

We use the initial condition to find b_n.

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx.$$

We are given a hint, that

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)dx = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2an}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

Then

$$b_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\,dx = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

Therefore in the sum we only have the even terms. The odd-indexed coefficients $b_{2m+1} = 0$, $m = 0, 1, 2, \ldots$; the even-indexed coefficients are, for $m = 1, 2, 3, \ldots$

$$b_{2m} = \frac{4(2m)}{\pi((2m)^2-1)} = \frac{8m}{\pi(4m^2-1)}.$$

Thus the concentration $C(x,t)$ is:

$$C(x,t) = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin\left(\frac{2m\pi x}{a}\right)\exp\left[-\left(\frac{4m^2\pi^2}{a^2}k + L\right)t\right].$$

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Problem 3: Find the Fourier Sine series of period 2π of the following function. Sketch the graph of the function to which the series converges (sketch at least three periods).

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 \le x \le \pi \end{cases}$$

SOLUTION:

Remember, the Fourier sine series of a function gives the odd periodic extension. That is, the function over $[0, L]$ is considered to be HALF of an odd function of which we want to take a periodic extension. The period of the odd periodic extension, of the Fourier sine series, is $2L$. Here, $L = \pi$ - which means that, when we calculate the Fourier sine series, it is automatically of period 2π!

We are asked to find a Fourier sine series of a function over $[0, \pi]$ ($L = \pi$). The Fourier sine series is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Since here $L = \pi$, our series will take the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \text{ where } b_n = \frac{2}{L}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

All that's left to calculate are the coefficients b_n.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2}(1)\sin(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(0)\sin(nx)\,dx \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\Big|_0^{\pi/2}\right] \\ b_n &= \frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]. \end{aligned}$$

This needs to be further dissected. First note that for n odd, $\cos(n\pi/2) = 0$. So let's consider the odd ($n = 2m-1$, $m = 1, 2, 3, \ldots$) and even indices ($n = 2m$, $m = 1, 2, 3, \ldots$) separately. That is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{m=1}^{\infty} b_{2m-1}\sin((2m-1)x) + \sum_{m=1}^{\infty} b_{2m}\sin(2mx).$$

Where

$$b_{2m-1} = \frac{2}{(2m-1)\pi},$$

$$\text{and } b_{2m} = \frac{2}{2m\pi}\left[1 - \cos\left(\frac{2m\pi}{2}\right)\right] = \frac{1}{m\pi}\left[1 - \cos(m\pi)\right] \Rightarrow b_{2m} = \frac{1}{m\pi}\left[1 - (-1)^m\right].$$

We can further simplify:

$$b_{2m} = \begin{cases} 0, & m \text{ even} \\ 2/m\pi, & m \text{ odd} \end{cases}$$

So again, consider odd and even indices separately,

$$\sum_{m=1}^{\infty} b_{2m}\sin(2mx) = \sum_{p=1}^{\infty} b_{2(2p)}\sin(2(2p)x) + \sum_{p=1}^{\infty} b_{2(2p-1)}\sin(2(2p-1)x) = \sum_{p=1}^{\infty} b_{4p}\sin(4px) + \sum_{p=1}^{\infty} b_{4p-2}\sin((4p-2)x),$$

where $b_{4p} = 0$ and $b_{4p-2} = 2/(2p-1)\pi$.

All together now, not writing in the zero terms,

$$f(x) = \sum_{m=1}^{\infty} b_{2m-1}\sin((2m-1)x) + \sum_{p=1}^{\infty} b_{4p-2}\sin((4p-2)x)$$

$$f(x) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin((2m-1)x) + \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{2p-1}\sin(2(2p-1)x).$$

This is sufficient. For the purposes of aesthetics only, let's change all the indices back to n

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2(2n-1)x)$$

and collect like terms, to obtain:

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\sin((2n-1)x) + \sin(2(2n-1)x)\right).$$

Finally we are asked to sketch the graph of the function to which the series converges - the odd periodic extension of $f(x)$. See Fig.4.

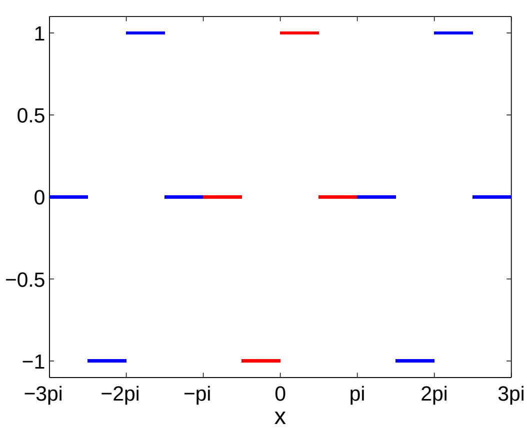

Figure 4: Three periods of the function the Fourier sine series of $f(x)$ converges to (aka the odd periodic extension of $f(x)$).