MATH 257/316 Assignment 6
Due: Fri, Nov 6 in class

Problem 1:

a) Compute the Fourier Sine Series of $f(x) = x^3 - x$ on $[0, 1]$.

b) Evaluate the series found in (a) at $x = 1/2$ to obtain

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots .$$

c) Drop all but the first two terms in the above series (i.e. leave just $1 - \frac{1}{3^3}$) and rearrange to obtain an approximation for the value of π. Use a calculator to write down this approximation to six significant figures.

d) Apply Parseval's identity to your answer in (a) to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

e) In the same fashion as for (c), drop all but the first two terms in the series given in (d) and rearrange to obtain an approximation for the value of π. Use a calculator to write down this approximation to six significant figures.

f) Which of the two approximations you have obtained most closely matches the value of π?

Problem 2:
The formula

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) ,$$

is a first order approximation to the derivative of f at x in terms of the value of f at x and $x-h$, where $h > 0$ is small. (We often use this to approximate u_t when numerically solving the heat equation.) By also using the values of f at $x-2h$, $x-3h$, $\ldots$, $x-nh$, we may derive a formula for $f'(x)$ that is accurate to $O(h^n)$. These are known as *backwards differentiation formulas* (BDFs).
By writing out the Taylor series of $f(x-h)$ and $f(x-2h)$ centred at x, show that

$$\frac{\frac{3}{2}f(x) - 2f(x-h) + \frac{1}{2}f(x-2h)}{h} = f'(x) + O(h^2) .$$

Problem 3:
Consider the following initial-boundary value problem for the heat equation (which corresponds to Example 15.3 from Lecture 19 of the notes with $L = 1$ and $g(x) = x$).

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \alpha^2 = 0.2 \\ BC &: \; u(0,t) = 1, \quad u_x(1,t) = 0 \\ IC &: \; u(x,0) = x \end{aligned}$$

Download the spreadsheet Heat0Slider0H.xls which has the Fourier Series programed into sheet2. Now implement the boundary conditions $u(0,t) = 1$ and $u_x(1,t) = 0$ in the finite difference scheme on sheet1 and compare your results to the Fourier Series solution. Plot the result and print out the solution at time $t = 0.588$ and hand this in.