Math 257/316 Assignment 7
Due Friday November 13th in class

Problem 1 (on steady-state solutions): Find the steady-state solutions of the heat equation $u_t = \alpha^2 u_{xx}$ that satisfy the following boundary conditions. For (a) and (b), interpret physically.
(a) $u(0,t) = -1$, $u(1,t) = 1$
(b) $u_x(0,t) = 2$, $u(1,t) = 0$
(c) $u_x(0,t) - 2u(0,t) = 0$, $u(1,t) = 1$.

Problem 2: Consider a bar of length $L = 1$ and thermal diffusivity $\alpha^2 = .2$ having initial temperature distribution given by

$$u(x,0) = \begin{cases} 1, & 0 \le x \le 1/2 \\ 0, & 1/2 < x \le 1. \end{cases}$$

Suppose that both ends of the bar are insulated - that is, $u_x(0,t) = u_x(1,t) = 0$.
(a) Use separation of variables to find the temperature $u(x,t)$.
(b) Determine the steady-state temperature in the bar.

Problem 3:

Consider heat radiation at a boundary - an insulated bar has an end exposed to some prescribed ambient temperature. How do we describe the flow of heat through the boundary? In other words, what's the boundary condition?

We assume that the rate heat flow across the boundary is proportional to the the difference between the temperature in the bar $u(x,t)$ and the ambient temperature U:

$$\text{heat flow} = \gamma(U - u),$$

where γ is some proportionality constant. This is called *Newton's law of cooling.* In one dimension, the heat flow is proportional to the rate of change in $u(x,t)$ with respect to x:

$$\text{heat flow} = -k\frac{\partial u}{\partial x}.$$

(think of it this way: the bigger the temperature gradient in a bar, the faster it will equalize). So

$$-k\frac{\partial u}{\partial x} = \gamma(U - u) \Rightarrow \frac{\partial u}{\partial x} - \frac{\gamma}{k}u = -\frac{\gamma U}{k}.$$

This is called a Robin boundary condition. Problem 1(c) gives a Robin condition on the left boundary, with $\gamma/k = 2$ and $U = 0$.

How do we treat this numerically? The Robin condition at the left endpoint of the bar $x = 0$, $u_x(0,t) - 2u(0,t) = 0$ can be approximated by the following difference quotient:

$$\begin{aligned} u_x(0,t) - 2u(0,t) &= \frac{u(0+\Delta x,t) - u(0-\Delta x,t)}{2\Delta x} - 2u(0,t) \\ 0 &= \frac{u(\Delta x,t) - u(-\Delta x,t)}{2\Delta x} - 2u(0,t) \end{aligned}$$

This equation reduces to the condition: $u(-\Delta x,t) = u(\Delta x,t) - 4\Delta x u(0,t)$. Now if $x_0 = 0$ is the left endpoint of the bar, then $x_0 - \Delta x = -\Delta x$, which falls outside the bar! However, we can trick the finite difference scheme into imposing this boundary condition by introducing a fictitious meshpoint $x_{-1} = x_0 - \Delta x$ and forcing the value of the solution $u(x_0 - \Delta x, t)$ at this point to be $u(\Delta x,t) - 4\Delta x u(0,t)$, in accordance with the condition above. That is, at each time t_k, $u_{-1}^k = u_1^k - 4\Delta x u_0^k$. You'll have to implement this BC the spreadsheet Heat_Assg7Prob3.xls.

Download Heat_Assg7Prob3.xls. This spreadsheet is set up to solve the heat equation $u_t = \alpha^2 u_{xx}$ with $\alpha^2 = 0.2$ on $0 \le x \le 1$ for $t \ge 0$. Apply boundary conditions as in 1(c), and initial condition:

$$u(x,0) = \begin{cases} 1 - 2x, & 0 \le x \le 1/2 \\ 2x - 1, & 1/2 < x \le 1 \end{cases},$$

and solve this problem numerically. Plot the solution at time $t = 0.1$, 0.2, 1, and 2 on the same page, print it out, and hand this in.