Math 257/316 Assignment 7
Due Friday November 13th in class
SOLUTIONS

Problem 1 (on steady-state solutions): Find the steady-state solutions of the heat equation $u_t = \alpha^2 u_{xx}$ that satisfy the following boundary conditions. For (a) and (b), interpret physically.
(a) $u(0,t) = -1$, $u(1,t) = 1$
(b) $u_x(0,t) = 2$, $u(1,t) = 0$
(c) $u_x(0,t) - 2u(0,t) = 0$, $u(1,t) = 1$.
Problem 2 SOLUTION:

In each case, assume $u(x,t) = u_\infty(x) + v(x,t)$, where $u_\infty(x)$ will give us our steady state solution. Briefly put, we are asked here to solve the ordinary differential equation $u''_\infty(x) = 0$ subject to the given boundary conditions. Our goal in finding a steady state solution is, in principle, to be able to write down a boundary value problem for $v(x,t)$ with homogeneous (aka zero) boundary conditions.
(a) We want to solve:

$$\begin{cases} u''_\infty(x) = 0 \\ u_\infty(0) = -1, \quad u_\infty(1) = 1. \end{cases}$$

Integrating twice, $u''_\infty(x) = 0 \Rightarrow u'_\infty(x) = A \Rightarrow u_\infty(x) = Ax + B$. Then applying the conditions:

$$\begin{aligned} -1 = u_\infty(0) = B \quad &\Rightarrow \quad B = -1 \\ 1 = u_\infty(1) = A + B \quad &\Rightarrow \quad A = -B + 1 \Rightarrow A = 2. \end{aligned}$$

Thus our steady state solution is

$$u_\infty(x) = 2x - 1.$$

Our physical interpretation is that as $t \to \infty$ the temperatures at the endpoints are as prescribed by the boundary conditions, with temperature at x=0 of -1 ($u_\infty(0) = -1$), and at x=1 of 1 ($u_\infty(1) = 1$), and the temperature profile between $x = 0$ and $x = 1$ is *linear*.
(b) We want to solve:

$$\begin{cases} u''_\infty(x) = 0 \\ u'_\infty(0) = 2, \quad u_\infty(1) = 0. \end{cases}$$

Integrating twice, $u''_\infty(x) = 0 \Rightarrow u'_\infty(x) = A \Rightarrow u_\infty(x) = Ax + B$. Then applying the conditions:

$$\begin{aligned} 2 = u'_\infty(0) = A \quad &\Rightarrow \quad A = 2 \\ 0 = u_\infty(1) = A + B \quad &\Rightarrow \quad B = -A \Rightarrow B = -2. \end{aligned}$$

Thus our steady state solution is

$$u_\infty(x) = 2(x - 1).$$

Our physical interpretation is that, as $t \to \infty$, we have a temperature on the right endpoint $x = 1$ of 0 (given by the boundary condition $u_\infty(1) = 0$) and the temperature at the left endpoint $x = 0$ is -2 ($u_\infty(0) = -2$), induced by the boundary condition on left decribing heat flow into the bar from outside the bar ($u_\infty(0) = 2$).
(c) We want to solve:

$$\begin{cases} u''_\infty(x) = 0 \\ u'_\infty(0) - 2u_\infty(0) = 0, \quad u_\infty(1) = 1. \end{cases}$$

Integrating twice, $u''_\infty(x) = 0 \Rightarrow u'_\infty(x) = A \Rightarrow u_\infty(x) = Ax + B$. Then applying the conditions:

$$\begin{aligned} 0 = u'_\infty(0) - 2u_\infty(0) = A - 2B \quad &\Rightarrow \quad A - 2B = 0 \\ 1 = u_\infty(1) = A + B \quad &\Rightarrow \quad A + B = 1. \end{aligned}$$

Solving simultaneously: $A - 2B = 0 \Rightarrow A = 2B$; then $A + B = 1$ means $3B = 1 \Rightarrow B = 1/3$. Since $A = 2B$, $A = 2/3$.

Thus our steady state solution is

$$u_\infty(x) = (2x + 1)/3.$$

Problem 2: Consider a bar of length $L = 1$ and thermal diffusivity $\alpha^2 = .2$ having initial temperature distribution given by

$$u(x,0) = \begin{cases} 1, & 0 \le x \le 1/2 \\ 0, & 1/2 < x \le 1. \end{cases}$$

Suppose that both ends of the bar are insulated - that is, $u_x(0,t) = u_x(1,t) = 0$.
(a) Use separation of variables to find the temperature $u(x,t)$.
(b) Determine the steady-state temperature in the bar.

Problem 2 SOLUTION:

$$\begin{aligned} u_t &= 0.2u_{xx},\ 0 \le x \le 1,\ t > 0\ (\alpha^2 = 0.2) \\ u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \\ u(x,0) &= \begin{cases} 1, & 0 \le x \le 1/2 \\ 0, & 1/2 < x \le 1. \end{cases} \end{aligned}$$

(a) Use separation of variables. Let $u(x,t) = X(x)T(t)$. Then $u_t = 0.2u_{xx}$ becomes $X(x)T'(t) = 0.2X''(x)T(t)$. We divide both sides by $0.2X(x)T(t)$ to obtain:

$$\frac{T'}{0.2T} = \frac{X''}{X} = \lambda, \tag{1}$$

where λ is a constant.

What happens to the boundary conditions under the separation of variables?

$$0 = u_x(0,t) = X(0)T(t) \Rightarrow X'(0) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

$$0 = u_x(1,t) = X'(1)T(t) \Rightarrow X'(1) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

So we have $X(0) = X'(\pi) = 0$.

Let's start with the T-equation from (1):

$$T'(t) = 0.2\lambda T(t),$$

which means $T(t) = Ce^{0.2\lambda t}$.

Next we deal with the X-equation in (1) with conditions $X(0) = X'(\pi) = 0$ derived from the boundary conditions

$$\begin{aligned} X'' &= \lambda X \\ X'(0) &= X'(2) = 0. \end{aligned}$$

This is our eigenvalue problem. There are 3 cases to consider: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

We begin with the $\lambda > 0$ case - recall from above that we expect this to only yield the trivial solution (aka $X = 0$). Set $\lambda = \mu^2 > 0$. Then $X''(x) - \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 - \mu^2 = 0$, which has roots $r = \pm\mu$.Thus $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X'(0) &= 0 \Rightarrow \mu C_1 - \mu C_2 = 0 \\ X'(1) &= 0 \Rightarrow \mu C_1 e^{\mu} - \mu C_2 e^{-\mu} = 0. \end{aligned}$$

Solving simultaneously we find $C_1 = C_2 = 0$. (The first equation gives $C_2 = C_1$, plugging into the first equation gives $\mu C_1 e^{\mu}(1 - e^{2\mu}) = 0$, which is only true if $C_1 = 0$ or $\mu = 0$. But $\lambda = \mu^2 > 0$ so μ is NOT zero by assumption, so $C_1 = 0$ and since $C_2 = C_1$, $C_1 = C_2 = 0$. You could also use $X(x) = \tilde{C_1}\sinh(\mu x) + \tilde{C_2}\cosh(\mu x)$, and would find $\tilde{C_1} = \tilde{C_2} = 0$.

All right, next we consider the $\lambda = 0$ case (we could consider it jointly with the $\lambda < 0$ or $\lambda > 0$ cases, if we're very careful, but for the purposes of a systematic approach we won't here). Then $X'' = 0 \Rightarrow X(x) =$

$Ax + B$. Applying boundary conditions, $0 = X'(0) = B \Rightarrow B = 0$; $0 = X'(1) = B \Rightarrow B = 0$. A remains arbitrary. So for the eigenvalue $\lambda_0 = 0$ we have a constant eigenfunction, $X_0 = A$.

Finally we look at the $\lambda < 0$ case. Set $\lambda = -\mu^2 < 0$. Then $X''(x) + \mu^2 X(x) = 0$ and $X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$ (for details see your textbook, section 3.3). We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X'(0) &= 0 \Rightarrow \mu C_1 \cos(0) + \mu C_2 \sin(0) = 0 \Rightarrow C_1 = 0 \\ X'(1) &= 0 \Rightarrow \mu C_2 \sin(\mu) = 0. \end{aligned}$$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \ldots$, the second condition tells us that $\mu_n = n\pi$ for $n = 1, 2, 3, \ldots$ Thus we have our eigenfunctions an eigenvalues for $\lambda < 0$:

$$\begin{aligned} \lambda_n &= -n^2\pi^2 \\ X_n(x) &= \cos(n\pi x). \end{aligned}$$

Now we re-assemble. Recall $u(x,t) = X(x)T(t)$. Therefore

$$u_n(x,t) = X_n(x)T_n(t) = \cos(n\pi x) \exp\left(-0.2\pi^2 n^2 t\right)$$

for $n = 1, 2, 3, \ldots$ are each solutions to the pde. The pde is linear so we can use the principle of superposition, and sum them to make up a general solution:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp\left(-0.2\pi^2 n^2 t\right),$$

where the b_n are constants.

We solve for the b_n using the initial condition. That is, $u(x, 0) = f(x)$ so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

which is a Fourier cosine series. Then

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = 2\int_0^1 f(x)\,dx \text{ and}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\mu_n x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx,\ n \neq 0$$

Now we are given $f(x)$, so

$$a_0 = 2\int_0^{1/2} (1)\,dx + 2\int_{1/2}^1 (0)\,dx = 1$$

$$\begin{aligned} a_n &= 2\int_0^{1/2} (1)\cos(n\pi x)\ dx + 2\int_{1/2}^1 (0)\cos(n\pi x)\,dx \\ &= \frac{2}{n\pi}\sin(n\pi x)\Big|_{x=0}^{x=1/2} \\ a_n &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

What are we to make of $\sin(n\pi/2)$?

n	1	2	3	4	5	6	7	8	...
$\sin(n\pi/2)$	1	0	-1	0	1	0	-1	0	

Let's split this into evens & odds - $n = 2m$ and $n = 2m-1$, m=1,2,3,... For $n = 2m$ even, $\sin(2m\pi/2) = 0$. For $n = 2m - 1$ odd,

m (n odd)	1 ($n=1$)	2 ($n=3$)	3 ($n=5$)	4 ($n=6$)	...
$\sin((2m-1)\pi/2)$	1	-1	1	-1	

$\Rightarrow \sin((2m-1)\pi/2) = (-1)^{m+1}$.

Our coefficents are therefore, for $m = 1, 2, 3, ...$

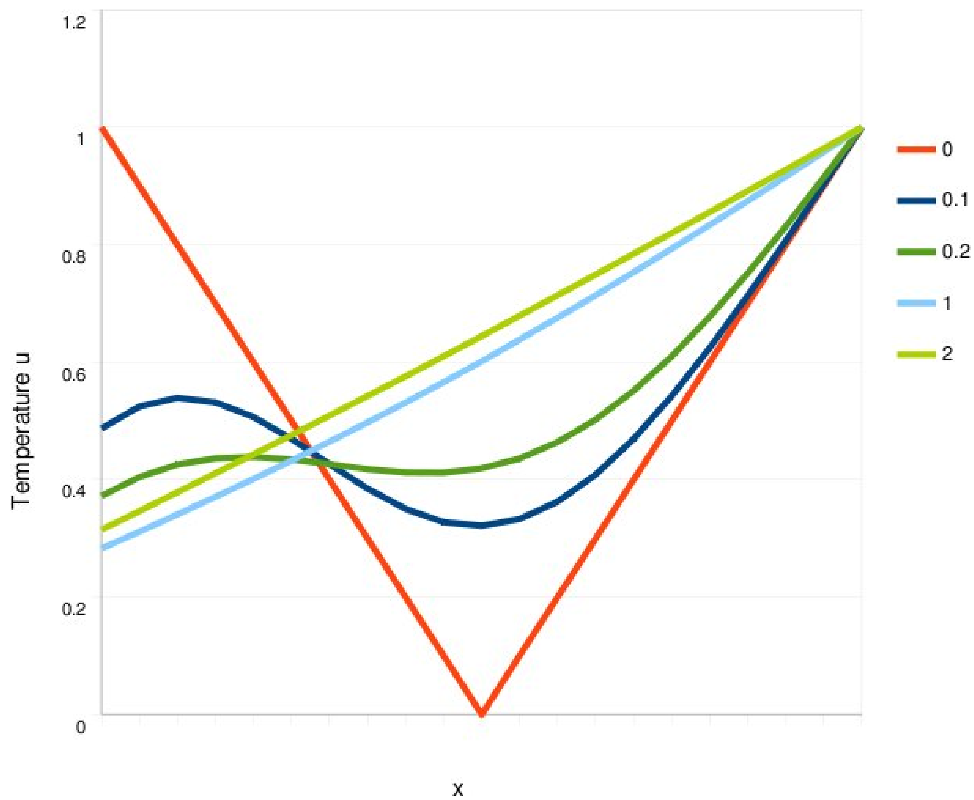
$$a_{2m} = 0,\ a_{2m-1} = \frac{2(-1)^{m+1}}{(2m-1)\pi}$$

Therefore our solution is:

$$u(x,t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_{2m} \cos(2m\pi x)\, e^{-0.2\pi^2(2m)^2 t} + \sum_{m=1}^{\infty} a_{2m-1} \cos((2m-1)\pi x)\, e^{-0.2\pi^2(2m-1)^2 t}$$

$$\Rightarrow u(x,t) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} \cos((2m-1)\pi x)\, e^{-0.2\pi^2(2m-1)^2 t}$$

Problem 3 SOLUTION: See spreadsheet for full solution. The plot that you turn in should look like:



Notice that as time advances the solution is converging to the steady-state solution that we calculated in 1(c), $u_\infty(x) = (2x-1)/3$.