**Math 257/316 Assignment 9**
**Due Monday November 30th in class**

**Problem 1:** Consider the wave equation for an infinite string

$$\begin{cases} u_{tt} = u_{xx}, & -\infty < x < \infty \\ u(x,0) = f(x), & u_t(x,0) = 0 \end{cases} \quad \text{where } f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the solution for $t = 0$, $t = 1/2$, and $t = 1$, and explain its behaviour as $t$ increases.

**Problem 2:** If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$.
(a) Write the initial/boundary value problem for the displacement $u(x,t)$ of an elastic string of length $L$, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x,0) = f(x)$.
(b) Find the general solution for this problem, using separation of variables.
(c) Find the displacement $u(x,t)$ of the elastic string if the initial position is given by

$$u(x,0) = f(x) = 2\sin\left(\frac{9\pi x}{2L}\right).$$

**Problem 3:** Consider the initial-boundary value problem

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 1 \\ u(0,t) = u(1,t) & = 0 \\ u(x,0) = \sin(2\pi x), & u_t(x,0) = 0 \end{cases}$$

(a) Use D'Alembert's solution to write down the solution of this pde.
(b) Verify that the solution at $t = 1/2, 1, 3/2, 2, ...$ is identical to the initial condition, $\sin(2\pi x)$. Give a physical interpretation as to why this should be.
(c) Now we solve this problem numerically! Download the spreadsheet *Wave_Assg9.xls*.
(i) Set the initial condition $u(x,0) = \sin(2\pi x)$, representing an initial displacement, exactly as before.
(ii) Set the left and right boundary conditions $u(0,t) = u(1,t) = 0$, exactly as before.
(iii) Now set the boundary condition representing the initial velocity of the string, $u_t(x,0) = 0$. This not as new as you think! The approach is similar to the derivative spatial boundary conditions we solved before. $u_t(x,0) = 0$ can be approximated by the following difference quotient (a backwards difference):

$$\begin{aligned} u_t(x,0) &= \frac{u(x,0) - u(x,0-\Delta t)}{\Delta t} \\ 0 &= \frac{u(x,0) - u(x,0-\Delta t)}{\Delta t}. \end{aligned}$$

This equation reduces to the condition: $u(x,-\Delta t) = u(x,0)$. But $-\Delta t < 0$ is outside of our range in time! However, we can trick the finite difference scheme into imposing this boundary condition by introducing a fictitious time point $t_{-1} = t_0 - \Delta t = -\Delta t$ and forcing the value of the solution $u(x,-\Delta t) = u(x,0)$ for all values of $x$, i.e. $u_n^{-1} = u_n^0$ at all meshpoints $x_n = n\Delta x$, $n = 0, 1, 2, 3, ...$ That is, you're going to add a ghost or fictitious point in time, corresponding to a row in your spreadsheet. Implement this initial condition (there's a blank row for you to fill in).
(iv) Derive the difference formula for the PDE $u_{tt} = 4u_{xx}$ and use it to fill the middle of the spreadsheet. Plot the solution at $t = 0.05, 0.1, 0.15.0.2, 0.25$ and turn that in.

**Problem 4:** Find the solution of the following BVP in the rectangle $\{(x,y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$. Note that the solution is unique only up to a constant!

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 \le x \le 1,\ 0 \le y \le 1 \\ u_x(0,y) = 0, & u_x(1,y) = \cos(\pi y) + \cos(2\pi y),\ 0 \le y \le 1 \\ u_y(x,0) = 0, & u_y(x,1) = 0,\ 0 \le x \le 1. \end{cases}$$

**Problem 5:** Find the solution of Laplace's equation in the semi-infinite strip $\{(x, y);\ 0 \le x \le 2,\ y \ge 0\}$ satisfying the following mixed boundary conditions:

$$\begin{cases} u(0,y)=0, & u_x(2,y)=0, \text{ for all } y \ge 0 \\ u(x,0)=2\sin(3\pi x/4)-3\sin(7\pi x/4) & \text{for all } 0 \le x \le 0, \\ \lim_{y\to+\infty} u(x,y)=0 \end{cases}$$

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**Additional problems - DO NOT TURN IN**

**Problem 6:** A vibrating string with fixed ends is governed by the initial-boundary value problem

$$\begin{cases} u_{tt}=u_{xx}, & 0<x<L,\ t>0 \\ u(0,t)=0, & u(L,t)=0, \\ u(x,0)=f(x), & u_t(x,0)=g(x). \end{cases}$$

(a) **The Plucked String**. If the string is lifted to a height $h_0$ at $x = a$ and released, the initial conditions are

$$f(x)=\begin{cases} h_0 x/a, & 0<x\le a \\ h_0(L-x)/(L-a), & a<x<L \end{cases}$$

and $g(x) = 0$. Find a formal solution to the plucked string problem.
(b) **The Struck String**. If the string is struck at $x = a$ and released, the initial conditions are $f(x) = 0$ and

$$g(x)=\begin{cases} v_0 x/a, & 0<x\le a \\ v_0(L-x)/(L-a), & a<x<L \end{cases}$$

where $v_0$ is a constant. Find a formal solution to the struck string solution.

**Problem 7 - The Telegraph Problem:** The telegraph equation describes the voltage and current on an electrical transmission line with distance and time. It's a model equation that demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line.

Use the method of separation of variables to derive a formal solution to the telegraph problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u &= \alpha^2\frac{\partial^2 u}{\partial x^2},\ 0<x<L,\ t>0 \\ u(0,t)=u(L,t)=0, &\quad t>0 \\ u(x,0)=f(x), &\quad \frac{\partial u}{\partial t}(x,0)=0,\ 0<x<L. \end{aligned}$$

**Problem 8:** Repeat problem 4 with different boundary conditions. Find the solution of the following BVP in the rectangle $\{(x, y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$:

$$\begin{cases} u_{xx}+u_{yy}=0, & 0\le x\le 1,\ 0\le y\le 1 \\ u_x(0,y)=0, & u_x(1,y)=\cos(\pi y)+\cos(2\pi y),\ 0\le y\le 1 \\ u_y(x,0)=\cos(3\pi x), & u_y(x,1)=0,\ 0\le x\le 1. \end{cases}$$

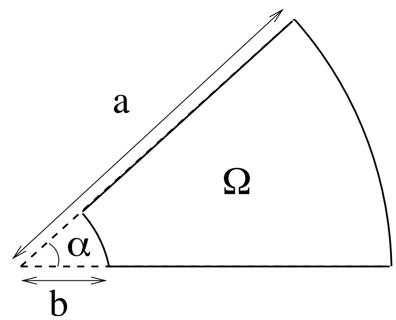
**Problem 8:** Find a solution to the following Dirichlet problem in the half annulus:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2} &= 0,\ \pi<r<2\pi,\ 0<\theta<\pi, \\ u(r,0)=\sin(r),\ u(r,\pi)=0, &\quad \pi\le r\le 2\pi \\ u(\pi,\theta)=u(2\pi,\theta)=0, &\quad 0\le\theta\le\pi. \end{aligned}$$

**Problem 9:** Find a solution to the following Neumann problem for an exterior domain:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} &= 0,\ r > 1,\ 0 < \theta < 2\pi, \\ \frac{\partial u}{\partial r}(1,\theta) = f(\theta), & \quad 0 \le \theta \le 2\pi \\ u(r,\theta)\,\text{remains bounded} \quad \text{as} & \quad \text{r} \to \infty. \end{aligned}$$

**Problem 10:** Consider a domain $\Omega$ obtained by taking a circular sector with angle $\alpha$ and radius $a$ and cutting out a smaller circular sector of radius $b$:




Find the solution of the following BVP in the domain $\Omega$:

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} &= 0 \text{ in } \Omega \\ v(r,0) = v(r,\alpha) = 0, \quad \text{for} & \quad b < r < a \\ v(b,\theta) = 0,\ v(a,\theta) = f(\theta) \quad \text{for} & \quad 0 < \theta < \alpha. \end{aligned}$$