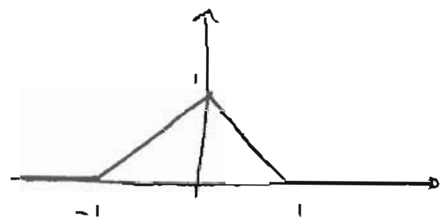
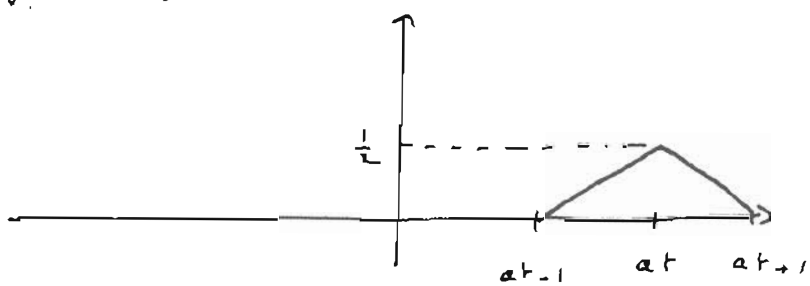
Problem 1 we have

$$u(x,t) = \frac{1}{2} f(x-at) + \frac{1}{2} f(x+at)$$

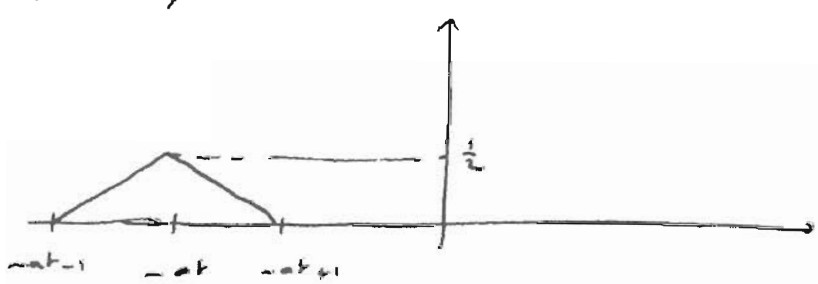
where f is given by



the function $\frac{1}{2} f(x-at)$ looks like.

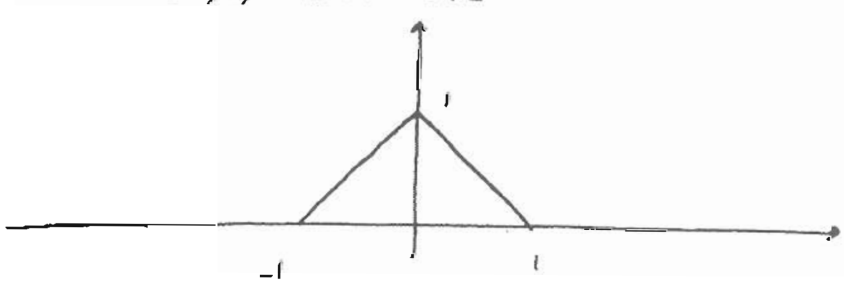


the function $\frac{1}{2} f(x+at)$ looks like



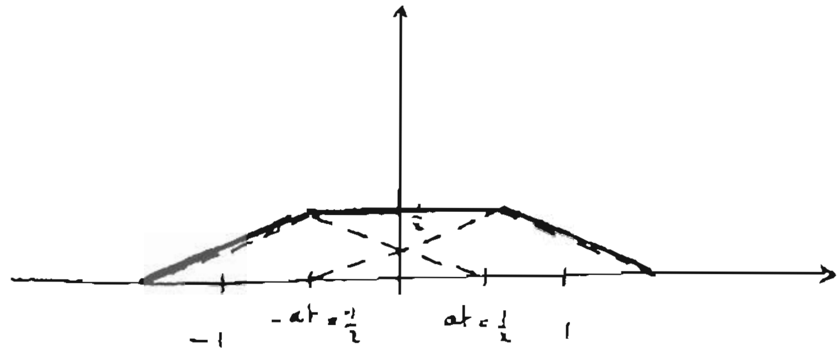
We deduce that $u(x,t)$ looks like:

for $t=0$

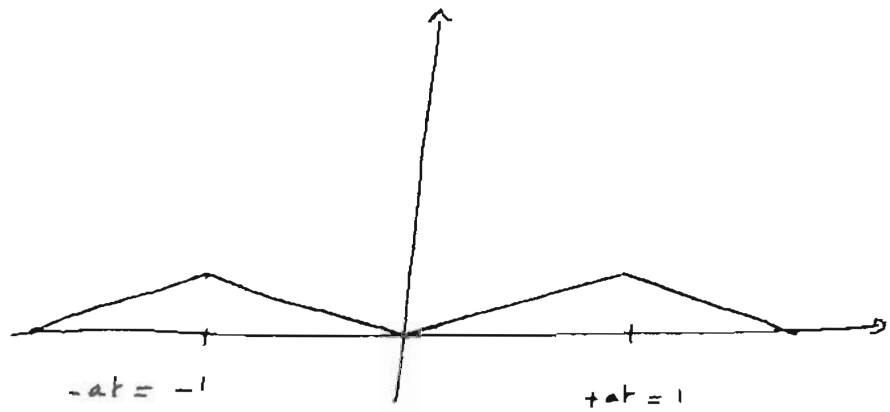


for $t = \frac{1}{2a}$

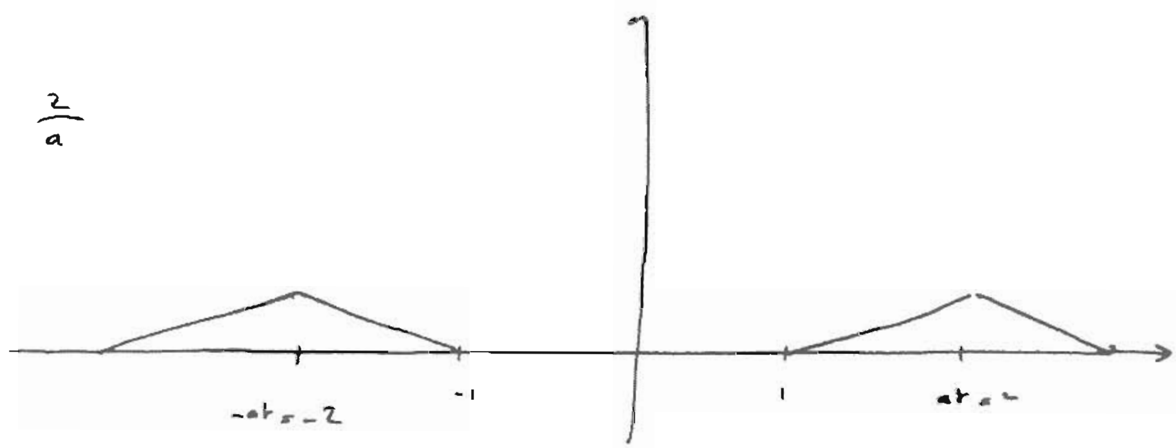
$(at = \frac{1}{2})$



for $t = \frac{1}{a}$



for $t = \frac{2}{a}$



(#2) $u(x,t)$ satisfies $\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < L \quad t > \\ u(0,t) = 0 \qquad u_x(L,t) = 0 & t \\ u(x,0) = f(x) \qquad u_t(x,0) = 0 & \end{cases}$

Since the B.C. are different from those considered in class, we need to go back to the Separation of variable methods:

we write $u(x,t) = X(x)T(t)$, then X and T must solve

$$X'' + \lambda X = 0 \quad , \quad X(0) = 0 \quad , \quad X'(L) = 0$$

$$T'' + a^2 \lambda T = 0 \qquad T'(0) = 0$$

the Eigenvalue pb for X has a non trivial solution if

$$\lambda_m = \left(\frac{(2m-1)\pi}{2L} \right)^2 \quad \text{and then} \quad X_m(x) = c \sin\left(\frac{(2m-1)\pi x}{2L} \right)^2$$

For a given λ_m, T must solve $T'' + a^2 \lambda_m T = 0$, and so

$$T(t) = C_1 \cos\left(\frac{a(2m-1)\pi}{2L} t \right) + C_2 \sin\left(\frac{a(2m-1)\pi}{2L} t \right)$$

the Condition $T'(0) = 0$ implies $C_2 = 0$

So the fundamental solutions are

$$u_n(x,t) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)\cos\left(\frac{a(2n-1)\pi}{2L}t\right)$$

As in class, we deduce that the displacement $u(x,t)$ is of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)\cos\left(\frac{a(2n-1)\pi}{2L}t\right)$$

(c) the c_n must be such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) = f(x) = 2\sin\left(\frac{9\pi}{2L}x\right)$$

So $c_5 = 2$ and $c_n = 0$ for all $n \neq 5$:

$$u(x,t) = 2\sin\left(\frac{9\pi}{2L}x\right)\cos\left(a\frac{9\pi}{2L}t\right)$$

3. $\begin{cases} u_{tt} = 4u_{xx}, & 0<x<1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \sin(2\pi x), & u_t(x,0) = 0 \end{cases}$

(a) Use D'Alembert's solution to write down the solution to this pde.

D'Alembert's solution is

$$u(x,t) = \frac{1}{2}\left[f_o(x+ct) + f_o(x-ct)\right] + \int_{x-ct}^{x+ct} g_o(s)\,ds$$

where c is the ~~...~~ wave speed, $f_o(x)$ is the odd periodic extension of the initial displacement ($u(x,0) = f(x)$), and $g_o(x)$ is the odd periodic extension of the initial velocity ($u_t(x,0) = g(x)$).

Here, $c = 2$ ($c^2 = 4$), $f(x) = \sin(2\pi x)$, $g(x) = 0$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\sin[2\pi(x+2t)] + \frac{1}{2}\sin[2\pi(x-2t)]}$$

($\sin(2\pi x)$ is an odd function)

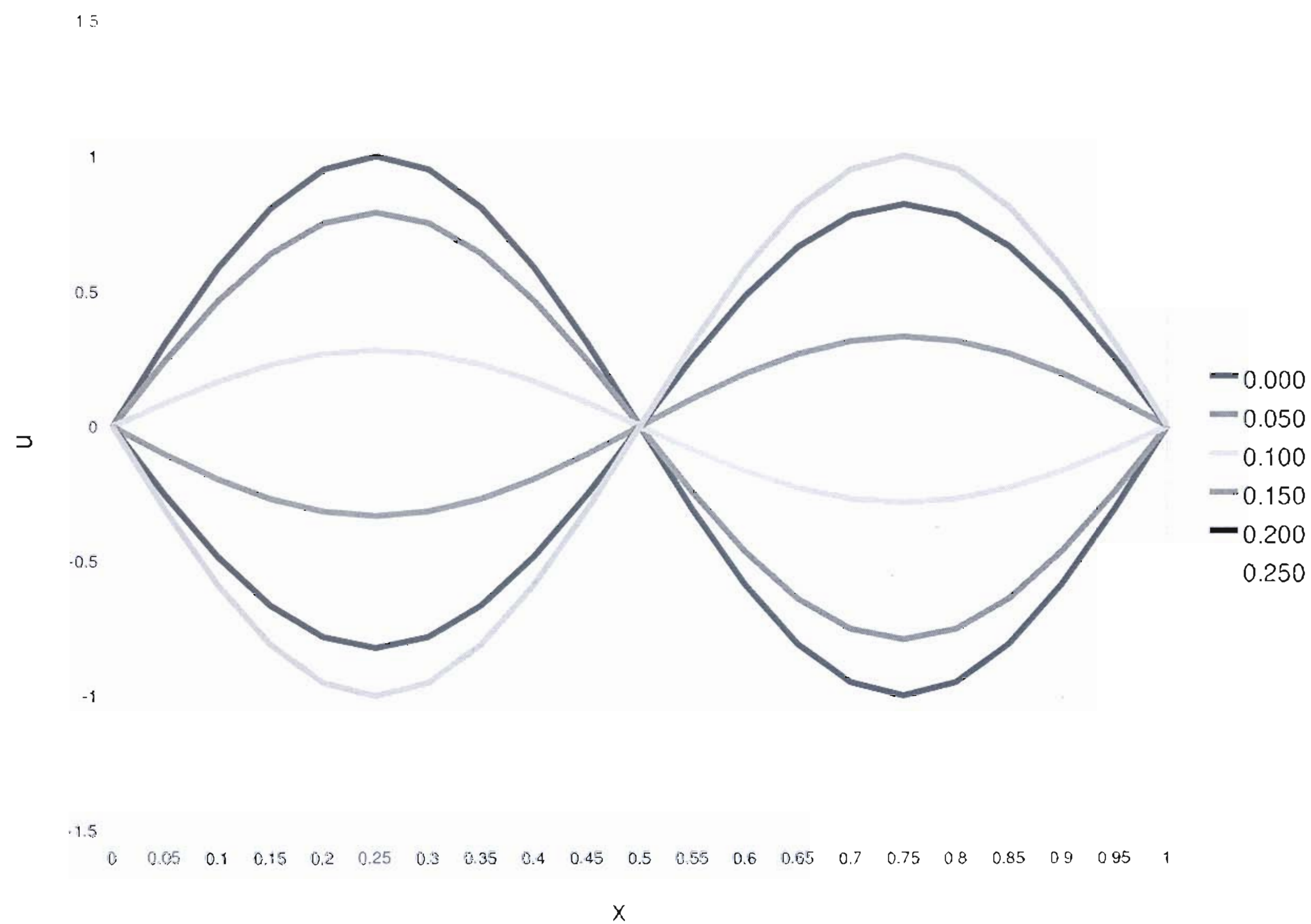
(b) Verify that the solution at $t = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ is identical to the initial condition $\sin(2\pi x)$

At $t = n/2$, $n = 1, 2, 3, \ldots$

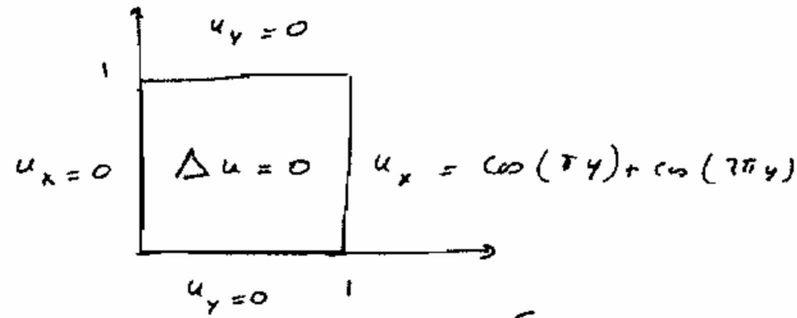
$$\begin{aligned} u(x, n/2) &= \tfrac{1}{2}\sin[2\pi(x + 2(\tfrac{n}{2}))] + \tfrac{1}{2}\sin[2\pi(x - 2(\tfrac{n}{2}))] \\ &= \tfrac{1}{2}\sin(2\pi x + 2n\pi) + \tfrac{1}{2}\sin(2\pi x - 2n\pi) \\ &= \tfrac{1}{2}\sin(2\pi x) + \tfrac{1}{2}\sin(2\pi x) \quad (\sin(\theta) \text{ is } 2\pi\text{-periodic}) \end{aligned}$$

$$\Rightarrow \boxed{u(x, n/2) = \sin(2\pi x), \text{ identical to the IC.}}$$

Why? The initial condition is the same as one of the standing wave modes of vibration! With no friction or gravity (etc) the standing wave will not decay.


1.5
1
0.5
0
-0.5
-1
-1.5
u
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6 0.65 0.7 0.75 0.8 0.85 0.9 0.95 1
X
0.000
0.050
0.100
0.150
0.200
0.250

Problem #4



the Separation equation are $\begin{cases} Y'' + \lambda Y = 0 & Y'(0) = Y'(1) = \\ X'' - \lambda X = 0 & X'(0) = 0 \end{cases}$

we deduce $\begin{cases} \lambda_n = (n\pi)^2 & n = 0, 1, 2 \ldots \\ Y_n(y) = \cos(n\pi y) \end{cases}$

the ODE for X gives $\begin{cases} X(x) = C_1 x + C_2 & \text{if } n = 0 \\ X(x) = C_1 \cosh(n\pi x) + C_2 \sinh(n\pi x) \end{cases}$

and $\begin{cases} X'(0) = C_1 = 0 & \text{if } n = 0 \\ X'(0) = n\pi C_2 = 0 & \text{so } C_2 = 0 \text{ if } n \neq 0 \end{cases}$

So $\begin{cases} X_n(x) = C_2 & \text{if } n = 0 \\ X_n(x) = C_1 \cosh(n\pi x) & \text{if } n \neq 0 \end{cases}$

the G.S. is
$$u(x,y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cosh(n\pi x) \cos(n\pi y)$$

which satisfies $u_x(1,y) = \sum_{n=1}^{\infty} c_n n\pi \sinh(n\pi) \cos(n\pi y)$

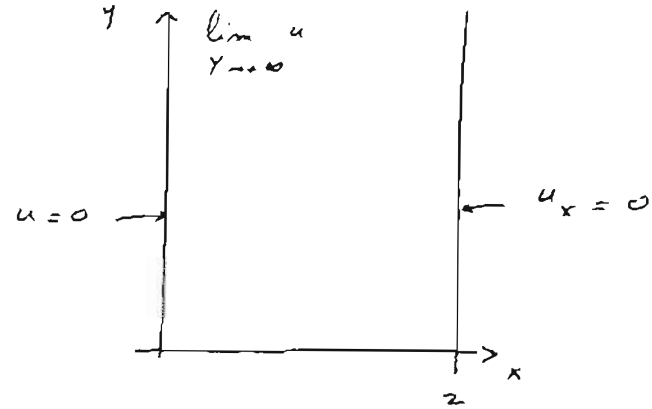
by identification, we see that we must have

$$c_1 \pi \sinh(\pi) = 1 \quad , \quad c_2 2\pi \sinh(2\pi) = 1, \quad c_n = 0 \quad n \neq 0,1,2$$

So
$$u(x,y) = \frac{c_0}{2} + \frac{1}{\pi \sinh(\pi)} \cosh(\pi x) \cos(\pi y) + \frac{1}{2\pi \sinh(2\pi)} \cosh(2\pi x) \cos(2\pi y)$$

(Note that there is no condition on c_0 : the solution is not unique).

Problem #5



$u(x,t) = X(x)\,Y(y)$ with

$$\begin{cases} X'' + \lambda X = 0 & X(0) = 0 \quad , \quad X'(2) = 0 \\ Y'' - \lambda Y = 0 & \lim_{y \to +\infty} Y(y) = 0 \end{cases}$$

the BVP for X has non trivial solutions if

$$\lambda_n = \left((2n-1)\frac{\pi}{4} \right)^2 \qquad n = 1, 2, \ldots$$

$$X_n(x) = k \sin\left((2n-1)\frac{\pi}{4} x \right)$$

For each λ_n, the eq. for Y gives

$$Y(x) = c_1 e^{(2n-1)\frac{\pi}{4} y} + c_2 e^{-(2n-1)\frac{\pi}{4} y}$$

and the condition $\lim_{y \to +\infty} Y(y) = 0$ yields $c_1 = 0$

So

$$Y_n(x) = k\, e^{-(2n-1)\frac{\pi}{4} y}$$

We deduce the following G.S. : $u(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{(2n-1)\pi}{4} y} \sin\left(\frac{2n-1}{4}\pi x\right)$

which satisfies $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{2n-1}{4}\pi x\right)$

by identification, we must take $c_2 = 2$, $c_4 = -3$, $c_n = 0$

the Solution of the BVP is thus

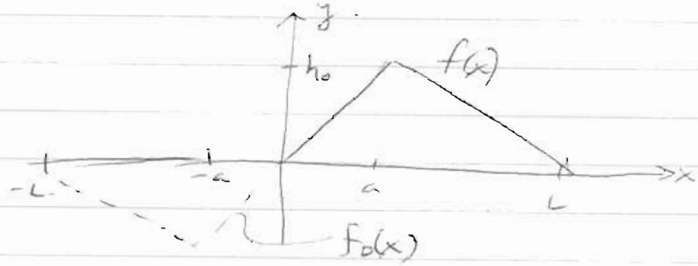
$$u(x,t) = 2\,e^{-\frac{3\pi}{4}y}\sin\left(\frac{3\pi}{4}x\right) - 3\,e^{-\frac{7\pi}{4}y}\sin\left(\frac{7\pi}{4}x\right)$$

Problem 6:

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

(a) $f(x) = \begin{cases} h_0 x / a & 0 < x < a \\ h_0 (L-x)/(L-a) & a \le x < L \end{cases}$

$g(x) = 0$



Since it's a string w/ fixed ends, we can use the D'Alembert solution:

$$u(x,t) = \frac{1}{2} f_0(x+t) + \frac{1}{2} f_0(x-t)$$

where $f_0(x)$ is the odd periodic extension of $f(x)$. (see plot)

To obtain the ~~odd~~ periodic extension – use the sine series! on $x \in [0, L]$

$$f_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \left\{ \int_0^a \frac{h_0 x}{a} \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^a \frac{h_0(L-x)}{L-a} \sin\left(\frac{n\pi x}{L}\right) dx \right\}$

$$\Rightarrow b_n = \frac{2 h_0 L (L-2a)}{a n^2 \pi^2 (L-a)} \sin\left(\frac{n\pi a}{L}\right) - \frac{4 h_0}{\pi n} \cos\left(\frac{n\pi a}{L}\right) + \frac{2 h_0 L}{n\pi (L-a)}$$

$$\boxed{\therefore\ u(x,t) = \sum_{n=1}^{\infty} \left[\left\{ \frac{h_0 L (L-2a)}{a^2 n^2 \pi^2 (L-a)} \sin\left(\frac{n\pi a}{L}\right) - \frac{2 h_0}{n\pi} \cos\left(\frac{n\pi a}{L}\right) + \frac{2 h_0 L}{n\pi (L-a)} \right\} \left\{ \sin\left(\frac{n\pi (x+t)}{L}\right) + \sin\left(\frac{n\pi (x-t)}{L}\right) \right\} \right]}$$

(b) $g(x) = \begin{cases} v_0 x/a, & 0 < x \leq a \\ v_0 (L-x)/(L-a), & a < x \leq L \end{cases}$, $f(x) = 0$.

Again, can use D'Alembert's solution!

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g_o(\xi)\, d\xi$$ where $g_o(\xi)$ is the odd periodic extension.

As in (a) (with v_0 instead of h_0)

$$g_o(\xi) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\xi}{L}\right), \quad b_n = \frac{2V_0 L}{n\pi(L-a)} - \frac{2V_0}{n\pi}\cos\left(\frac{n\pi a}{L}\right) + \frac{2V_0 L(L-2a)}{a^2 n^2\pi^2 (L-a)}\sin\left(\frac{n\pi a}{L}\right)$$

Then $$\frac{1}{2}\int_{x-t}^{x+t} g_o(\xi)\, d\xi = \frac{1}{2}\sum_{n=1}^{\infty} b_n \int_{x-t}^{x+t} \sin\left(\frac{n\pi\xi}{L}\right) d\xi = \frac{-1}{2}\sum_{n=1} \frac{L}{n\pi} b_n \cos\left(\frac{n\pi\xi}{L}\right)\Bigg|_{\xi=x-t}^{x+t}$$

$$= \sum_{n=1}^{\infty} \frac{L}{2n\pi} b_n \left[\cos\left(\frac{n\pi(x-t)}{L}\right) - \cos\left(\frac{n\pi(x+t)}{L}\right)\right]$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{L}{2n\pi}\left[\frac{2v_0 L}{n\pi(L-a)} - \frac{2v_0}{n\pi}\cos\left(\frac{n\pi a}{L}\right) + \frac{2v_0 L(L-2a)}{a^2 n^2 \pi^2 (L-a)}\sin\left(\frac{n\pi a}{L}\right)\right] \cdot \left[\cos\left(\frac{n\pi(x-t)}{L}\right) - \cos\left(\frac{n\pi(x+t)}{L}\right)\right]$$

Problem 7 $\begin{cases} u_{tt} + u_t + u = \alpha^2 u_{xx} \\ u(0,t) = u(L,t) = 0, \quad t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = 0 \end{cases}$

Use separation of variables : $u(x,t) = X(x)T(t)$

eq. becomes $\quad XT'' + XT' + XT = \alpha^2 X'' T$

$$\div \alpha^2 XT \Rightarrow \left(\frac{T'' + T' + T}{\alpha^2 T}\right) = \frac{X''}{X} = \lambda, \text{ const.}$$

For $\underline{\lambda = -\mu^2 < 0}$ $\quad \begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(L) = 0 \end{cases} \Rightarrow$ eigenfunctions $X_n(x) = \sin(n\pi x/L)$

eigenvalues $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$

($\lambda \geq 0$ yields no eigenvalues / eigenfunctions)

Then the T-equation becomes

$$T'' + T' + T = \alpha^2 \lambda_n T$$

or $\begin{cases} T'' + T' + (1 - \alpha^2\lambda_n) T = 0 \\ T'(0) = 0 \end{cases}$ where

Assume $T = e^{rt} \Rightarrow$ char. eq. $\quad r^2 + r + (1 - \alpha^2\lambda_n) = 0$

so $\underline{r = \frac{-1 \pm \sqrt{1 - 4(1 - \alpha^2\lambda_n)}}{2}} = -\frac{1}{2} \pm i\,\frac{\sqrt{3 - \alpha^2\lambda_n 4}}{2}$ (Note: $\lambda_n < 0$! above)

$$\therefore \; T_n(t) = C_1 e^{-t/2} \sin\left(\frac{\sqrt{3 - 4\alpha^2\lambda_n}}{2}\, t\right) + C_2\, e^{-t} \cos\left(\frac{\sqrt{3 - \alpha^2\lambda_n}}{2}\, t\right)$$

$T_n'(0) = 0 \Rightarrow \quad C_2 = C_1\sqrt{3 - 4\alpha^2\lambda_n}$

and $T_n(t) = C_1 e^{-t}\left\{ \sin\left(\frac{\sqrt{3-4\alpha^2\lambda_n}}{2}\, t\right) + \sqrt{3 - 4\alpha^2\lambda_n}\, \cos\left(\frac{\sqrt{3-4\alpha^2\lambda_n}}{2}\, t\right)\right.$

Then since $u(x,t) = X(x)T(t)$ & eq. is linear (use superposition)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-t}\left[\sin\left(\frac{\sqrt{3-4\alpha^2\lambda_n}}{2}\, t\right) + \sqrt{3-4\alpha^2\lambda_n}\, \cos\left(\frac{\sqrt{3-4\alpha^2\lambda_n}}{2}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Now use $u(x,0) = f(x)$ to find C_n:

$$f(x) = \sum_{n=1}^{\infty} c_n \sqrt{3-4\alpha^2\lambda_n}\, \sin\left(\frac{n\pi x}{L}\right) \quad \ldots \text{ a sine series!}$$

$$\Rightarrow \quad \sqrt{3-4\alpha^2\lambda_n}\, c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{or} \quad c_n = \frac{2}{L\sqrt{3-4\alpha^2\lambda_n}}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Thus,

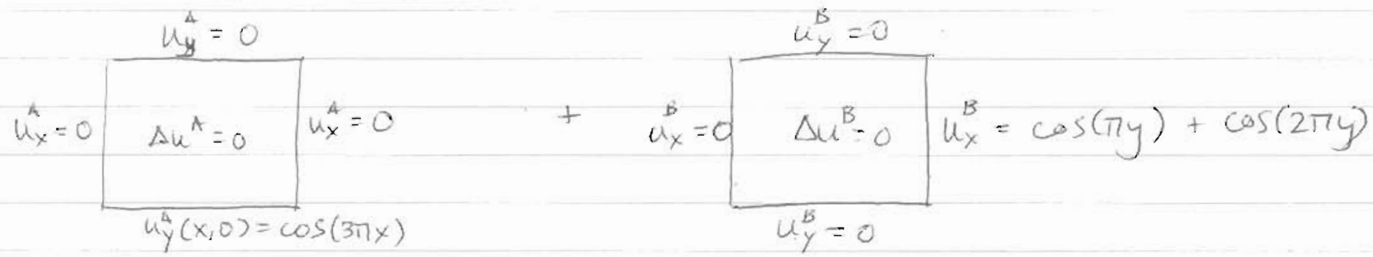
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-t/2}\left[\sin\left(\frac{\sqrt{3-4\alpha^2\lambda_n}}{2}t\right) + \cos\left(\frac{\sqrt{3-\alpha^2\lambda_n}}{2}t\right)\cdot\sqrt{3-4\alpha^2\lambda_n}\right]\sin\left(\frac{n\pi x}{L}\right).}$$

$$\text{where } \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad c_n = \frac{2}{L\sqrt{3-4\alpha^2\lambda_n}}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

These are damped oscillations; $u \to 0$ as $t \to \infty$.

Problem 8 : $\begin{cases} \Delta u = 0 & , 0 \le x \le 1, \quad 0 \le y \le 1 \\ u_x(0,y) = 0 , & u_x(1,y) = \cos(\pi y) + \cos(2\pi y) \\ u_y(x,0) = \cos(3\pi x) , & u_y(x,1) = 0 \end{cases}$

To use separation of variables, need $\underline{\underline{2}}$ homogeneous BCs in either x or y.
Exploit linearity of equation and split into 2 sub-problems



So $u(x,t) = u^A(x,t) + u^B(x,t)$ where

$$\begin{cases} \Delta u^A = 0 , & 0 \le x \le 1, 0 \le y \le 1 \\ u_x^A(0,y) = u_x^A(1,y) = 0 \\ u_y^A(x,0) = \cos(3\pi x), & u_y^A(x,1) = 0 \end{cases} \quad \text{(A)}$$

&

$$\begin{cases} \Delta u^B = 0, & 0 \le x, y \le 1 \\ u_x^B(0,y) = 0, & u_x^B(1,y) = \cos(\pi y) + \cos(2\pi y) \\ u_y^B(x,0) = u_y^B(x,1) = 0 \end{cases} \quad \text{(B)}$$

Problem (B) is the same as Assignment # problem 4.
(see solutions)

We'll solve (A):
separation of variables to obtain: $\begin{cases} X'' - \lambda X = 0, & X'(0) = X'(1) = 0 \\ Y'' + \lambda Y = 0, & Y'(\ ) = 0 \end{cases}$

As in assignment problem 3 (with X & Y reversed)

$$\begin{cases} \lambda_n = -(n\pi)^2 \\ X_n(x) = \cos(n\pi x) , \quad n = 0, 1, 2, \cdots \\ \cdots \end{cases}$$

For $\lambda = 0$: $Y_0'' = 0 \Rightarrow Y_0 = C_1 y + C_2$; $0 = Y_0'(1) \Rightarrow C_1 = 0$ so $Y_0 = \text{const}$

For $\lambda = -n^2\pi^2, n \ne 0$: $Y_n'' - n^2\pi^2 Y = 0 \Rightarrow Y_n = C_1 \cosh(n\pi y) + C_2 \sinh(n\pi y)$

$Y_n'(1) = 0 \Rightarrow n\pi C_1 \sinh(n\pi) + n\pi C_2 \cosh(n\pi) = 0$

so $C_2 = -C_1 \tanh(n\pi)$

and $Y_n = C_1 \cosh(n\pi y) - C_1 \tanh(n\pi) \sinh(n\pi y)$

$$Y_n = \underbrace{\frac{C_1}{\cosh(n\pi)}}_{\tilde{C}_n}\left[\cosh(n\pi)\cosh(n\pi y) - \sinh(n\pi)\sinh(n\pi y)\right]$$

$$\Rightarrow \boxed{Y_n = \tilde{C}_n \cosh[n\pi(y-1)]}$$

Thus,

$$u^A(x,y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cosh[n\pi(y-1)]\cos(n\pi x)$$

use condition $u^A_y(x,0) = \cos(3\pi x)$ to obtain C_n:

$$u^A_y(x,0) = \sum_{n=1}^{\infty} -n\pi C_n \sinh(n\pi)\cos(n\pi x)$$

$$\cos(3\pi x) = \qquad \parallel$$

$$\Rightarrow \quad -n\pi C_n \sinh(n\pi) = 2\int_0^1 \cos(3\pi x)\cos(n\pi x)\,dx$$

$$\text{so } \quad C_n = 0,\ n \neq 3\ ; \quad C_3 = \frac{-\cancel{2}\,1}{3\pi\sinh(3\pi)}$$

Note: $2\int_0^1 \cos(3\pi x)dx = 0$, the net flux through the boundary is zero, so a solution exists!

$$\Rightarrow \boxed{u^A(x,y) = C_0 - \frac{1}{3\pi\sinh(3\pi)}\cosh[3\pi(y-1)]\cos(3\pi x)}$$

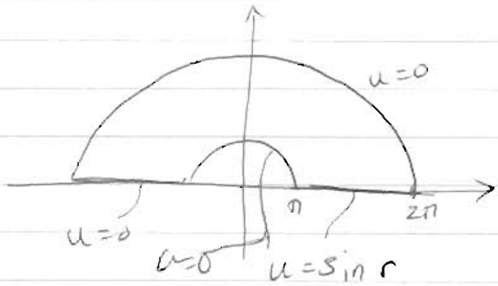
and $u(x,y) = u^A(x,y) + u^B(x,y)$ (sol. to u^B in Problem 4)

$$u(x,y) = A - \frac{1}{3\pi\sinh(3\pi)}\cosh[3\pi(y-1)]\cos(3\pi x) + \frac{1}{\pi\sinh(\pi)}\cosh(\pi x)\cos(\pi y) + \frac{1}{2\pi\sinh(2\pi)}\cosh(2\pi x)\cos(2\pi y)$$

where A is an arbitrary constant (solution is not unique)

Problem 8 (second one)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & \pi < r < 2\pi. \\ u(r,0) = \sin(r), \quad u(r,\pi) = 0 \\ u(\pi,\theta) = u(2\pi,\theta) = 0 \end{cases}$$



Use separation of variables: $u(r,\theta) = R(r)\Theta(\theta)$

obtain $\begin{cases} r^2R'' + rR' - \lambda R = 0, & R(\pi) = R(2\pi) = 0 \\ \Theta'' + \lambda\Theta = 0, & \Theta(\pi) = 0 \end{cases}$

Since we have 2 homog. BCs in r, look at R eq. first.

Let $R = r^\gamma \Rightarrow$ char eq $\gamma(\gamma-1) + \gamma - \lambda = 0$ or $\gamma^2 = \lambda$.

Case $\lambda = -\mu^2 < 0$: $\gamma^2 = -\mu^2$ so $\gamma = i\mu$. and $R(r) = C_1\cos(\mu\ln(r)) + C_2\sin(\mu\ln(r))$

$R(\pi) = 0 \Rightarrow C_1\cos(\mu\ln(\pi)) + C_2\sin(\mu\ln(\pi)) \Rightarrow C_1 = -C_2\tan(\mu\ln(\pi))$.

so $R(r) = C_2[-\tan(\mu\ln\pi)\cos(\mu\ln(r)) + \sin(\mu\ln(r))]$

$= \frac{C_2}{\cos(\mu\ln\pi)}[-\sin(\mu\ln\pi)\cos(\mu\ln r) + \cos(\mu\ln\pi)\sin(\mu\ln r)]$

$C_\mu \leftarrow$

$= C_\mu\sin(\mu\ln r - \mu\ln\pi)$ so $\boxed{R(r) = C_\mu\sin\left[\mu\ln\left(\frac{r}{\pi}\right)\right]}$

$R(2\pi) = 0 \Rightarrow C_\mu\sin[\mu\ln(2)] = 0 \Rightarrow \mu\ln(2) = n\pi \Rightarrow \mu = \frac{n\pi}{\ln(2)}$

Thus for $\lambda < 0$, eigenvalues $\boxed{\lambda_n = -\left(\frac{n\pi}{\ln 2}\right)^2}$

eigenfunctions, $\boxed{R_n = C_n\sin\left[\frac{n\pi\ln(r/\pi)}{\ln(2)}\right]}$

Case $\lambda = 0$: $\gamma^2 = 0$ so $\gamma = 0$ and $R(r) = C_1 + C_2\ln(r)$

$R(\pi) = 0 = C_1 + C_2\ln(\pi) \Rightarrow C_1 = -C_2\ln(\pi)$

and $R(r) = C_1[-\ln(\pi) + \ln(r)] = C_1\ln(r/\pi)$

$R(2\pi) = 0 = C_1\ln\left(\frac{2\pi}{\pi}\right) = C_1\ln(2) \Rightarrow C_1 = 0$ $\therefore$ no eigenfunction/ zero eigenvalue.

Case $\lambda > 0$: $\lambda = \mu^2 > 0 \Rightarrow \gamma^2 = \mu^2$, $\gamma = \pm\mu$ and $R(r) = C_1 r^{\mu} + C_2 r^{-\mu}$

$R(\pi) = 0 \Rightarrow C_1 \pi^{\mu} + C_2 \pi^{-\mu}$ so $C_2 = -C_1 \pi^{2\mu}$

and then $R(r) = C_1 r^{\mu} - C_1 \pi^{2\mu} r^{-\mu}$

$= C_1 \left[\left(\frac{r}{\pi}\right)^{\mu} + \left(\frac{r}{\pi}\right)^{-\mu} \right]$

$R(2\pi) = 0 \Rightarrow C_1 [2^{\mu} + 2^{-\mu}] = 0 \Rightarrow C_1 = 0$

$\boxed{\therefore \text{ no eigenfunctions/eigenvalues for } \lambda > 0}$

Thus we have $\boxed{\lambda_n = -\left(\frac{n\pi}{\ln(2)}\right)^2, \quad R_n(r) = \sin\left[\frac{n\pi \ln(r/\pi)}{\ln(2)}\right]}$

Now for the Θ-eq: $\begin{cases} \Theta_n'' - \left(\frac{n\pi}{\ln(2)}\right)^2 \Theta = 0 \\ \Theta(\pi) = 0 \end{cases}$

obtain $\Theta(\theta) = C_1 \sinh\left(\frac{n\pi\theta}{\ln(2)}\right) + C_2 \cosh\left(\frac{n\pi\theta}{\ln(2)}\right)$

$\Theta(\pi) = 0 \Rightarrow$ $\boxed{\Theta_n = \sinh\left(\frac{n\pi}{\ln(2)}(\theta - \pi)\right)}$

Then by superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi \ln(r/\pi)}{\ln(2)}\right] \sinh\left(\frac{n\pi(\theta-\pi)}{\ln(2)}\right)$$

Now, $u(r,0) = \sin(r) = \sum_{n=1}^{\infty} -C_n \sinh\left(\frac{n\pi^2}{\ln 2}\right) \cdot \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right)$

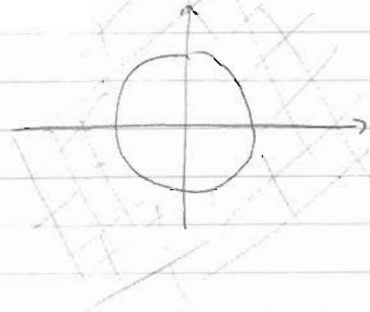
a sine series! so $-C_n \sinh\left(\frac{n\pi^2}{\ln 2}\right) = \frac{2}{\pi}\int_{\pi}^{2\pi} \sin r \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) dr$.

or $C_n = \frac{-2}{\pi \sinh(n\pi^2/\ln 2)} \int_{\pi}^{2\pi} \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) \sin(r)\, dr$

$$\boxed{\begin{aligned} &\text{Thus, } u(r,\theta) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) \sinh\left(\frac{n\pi(\theta-\pi)}{\ln(2)}\right) \\ &\text{where } C_n = \frac{-2}{\pi \sinh(n\pi^2/\ln(2))} \int_{\pi}^{2\pi} \sin(r) \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) dr \end{aligned}}$$

Problem 9: $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$u_r(1,\theta) = f(\theta)$, $|u| < \infty$ as $r \to \infty$



Use separation of variables $u(r,\theta) = R(r)\Theta(\theta)$

Obtain $\begin{cases} r^2 R'' + rR' - \lambda R = 0, & R \text{ bded as } r \to \infty \\ \Theta'' + \lambda \Theta = 0, & \text{periodic BCs. } \Theta(\theta + 2\pi) = \Theta(\theta) \end{cases}$

Θ-eq: since BCs are periodic, sol'n has to be, so $\lambda \geq 0$.

$\lambda = 0$: $\Theta'' = 0 \Rightarrow \Theta = C_1\theta + C_2$.
since $\Theta(2\pi + \theta) = \Theta(\theta)$, $C_1 = 0 \Rightarrow \begin{cases} \lambda_0 = 0 \\ \Theta_0 = 1 \end{cases}$

$\lambda = \mu^2 > 0$. $\Theta'' + \mu^2 \Theta = 0 \Rightarrow \Theta(\theta) = C_1 \cos(\mu\theta) + C_2 \sin(\mu\theta)$
but $\Theta(2\pi + \theta) = \Theta(\theta)$; use $\theta = -\pi$ to find C_1, C_2.

$\Theta(\pi) = \Theta(-\pi)$

$\Rightarrow C_1 \cos(\mu\pi) + C_2 \sin(\mu\pi) = C_1 \cos(-\mu\pi) + C_2 \sin(-\mu\pi)$

$\Rightarrow 2C_2 \sin(\mu\pi) = 0$ as $\sin(-\varphi) = -\sin\varphi$, $\cos(-\varphi) = \cos\varphi$

$\sin\varphi$ has roots at $\varphi = n\pi$, $n = 1, 2, \ldots$

so $\mu\pi = n\pi$

$\boxed{\mu = n}$

Thus $\lambda_n = +n^2$

$\Theta_n = a_n \cos(n\theta) + b_n \sin(n\theta)$

R-eq: $r^2 R'' + rR' - n^2 R = 0$ as $\lambda_n = n^2$

let $R = r^\gamma \Rightarrow$ char. eq. $\gamma(\gamma - 1) + \gamma - n^2 = 0$

$\gamma = \pm n$.

So $R(r) = C_1 r^n + C_2 r^{-n}$.

but R bounded as $r \to \infty \Rightarrow C_1 = 0$ So $\underline{R_n = r^{-n}}$

Then by superposition

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n} \{A_n \cos(n\theta) + B_n \sin(n\theta)\}$$

Find constants using BC $u_r(1, \theta) = f(\theta)$:

$$u_r(1,\theta) = \sum_{n=1}^{\infty} \{-nA_n \cos(n\theta) - nB_n \sin(n\theta)\}$$

complete Fourier series, with missing $n=0$ term!

so for there to be a solution, $\int_0^{2\pi} f(\theta)\, d\theta = 0$.

Assuming that's the case, $A_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta$

$$B_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta$$

Thus, assuming $\int_0^{2\pi} f(\theta)\, d\theta = 0$,

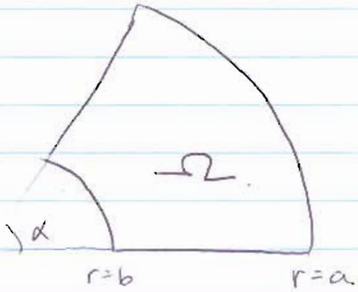
$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

where A_0 is <u>arbitrary</u>, $A_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta$

$$B_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta.$$

Problem 10:

$$\begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \\ v(r,0) = v(r,\alpha) = 0 \\ v(b,\theta) = 0, \; v(a,\theta) = f(\theta) \end{cases}$$



Use separation of variables $v(r,\theta) = R(r)\Theta(\theta)$

Obtain $\begin{cases} r^2 R'' + rR' - \lambda R = 0 & ; R(b) = 0 \\ \Theta'' + \lambda \Theta = 0 & ; \Theta(0) = \Theta(\alpha) = 0 \end{cases}$

Θ-eq first: $\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(\alpha) = 0 \end{cases} \Rightarrow \begin{cases} \Theta_n = \sin\left(\frac{n\pi\theta}{\alpha}\right) \\ \lambda_n = +\left(\frac{n\pi}{\alpha}\right)^2 \end{cases}$ (see class notes)

R-eq: $\begin{cases} r^2 R'' + rR' + \left(\frac{n\pi}{\alpha}\right)^2 R = 0 \\ R(b) = 0 \end{cases}$

Let $R = r^\gamma \Rightarrow$ char. eq. $\gamma(\gamma-1) + \gamma - \left(\frac{n\pi}{\alpha}\right)^2 = 0 \Rightarrow \gamma = \pm \frac{n\pi}{\alpha}$.

so $R(r) = C_1 r^{n\pi/\alpha} + C_2 r^{-n\pi/\alpha}$.

$R(b) = 0 = C_1 b^{n\pi/\alpha} + C_2 b^{-n\pi/\alpha} \Rightarrow C_2 = -C_1 b^{2n\pi/\alpha}$.

so $R(r) = C_1\left(r^{n\pi/\alpha} - b^{2n\pi/\alpha} r^{-n\pi/\alpha}\right) = \overbrace{C_1 b^{n\pi/\alpha}}^{\tilde{C}_n}\left(r^{n\pi/\alpha} b^{-n\pi/\alpha} - r^{-n\pi/\alpha} b^{n\pi/\alpha}\right)$

$$\Rightarrow \boxed{R_n(r) = C_n\left[\left(\frac{r}{b}\right)^{n\pi/\alpha} - \left(\frac{r}{b}\right)^{-n\pi/\alpha}\right]}$$

Then by superposition $v(r,\theta) = \sum_{n=1}^{\infty} C_n\left[\left(\frac{r}{b}\right)^{n\pi/\alpha} - \left(\frac{r}{b}\right)^{-n\pi/\alpha}\right] \sin\left(\frac{n\pi\theta}{\alpha}\right)$

Now $v(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} C_n\left[\left(\frac{a}{b}\right)^{n\pi/\alpha} - \left(\frac{a}{b}\right)^{-n\pi/\alpha}\right] \sin\left(\frac{n\pi\theta}{\alpha}\right)$, a sine series

$$\Rightarrow C_n = \frac{2}{\alpha\left[\left(\frac{a}{b}\right)^{n\pi/\alpha} - \left(\frac{a}{b}\right)^{-n\pi/\alpha}\right]} \int_0^\alpha f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta$$

$\longrightarrow$ flip

Note that we can write:

$$\left(\frac{r}{b}\right)^{n\pi/\alpha} = \exp\left[\ln\left(\left(\frac{r}{b}\right)^{n\pi/\alpha}\right)\right] = \exp\left[\frac{n\pi}{\alpha}\ln(r/b)\right]$$

$$\left(\frac{r}{b}\right)^{-n\pi/\alpha} = \exp\left[-\frac{n\pi}{\alpha}\ln(r/b)\right]$$

so $$\left(\frac{r}{b}\right)^{n\pi/\alpha} - \left(\frac{r}{b}\right)^{-n\pi/\alpha} = 2\sinh\left[\left(\frac{n\pi}{\alpha}\right)\ln(r/b)\right]$$

<u>so</u>

$$\boxed{\begin{aligned} v(r,\theta) &= \sum_{n=1}^{\infty} 2c_n \sinh\left[\frac{n\pi}{\alpha}\ln(r/b)\right]\sin\left(\frac{n\pi\theta}{\alpha}\right) \\ \text{where}\quad c_n &= \frac{1}{\alpha\sinh\left[\frac{n\pi}{\alpha}\ln(a/b)\right]}\int_0^{\alpha} f(\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)d\theta \end{aligned}}$$