1. $y'' + \lambda y = 0$ $\qquad y(0) = 0$

$\qquad\qquad\qquad\qquad\qquad y'(5) = 0$

Case 1) $\lambda < 0$

$$\Rightarrow \quad y(x) = a_1 e^{\sqrt{-\lambda}x} + a_2 e^{-\sqrt{-\lambda}x}$$

$$y(0) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

$$y'(x) = a_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}x} - a_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}x}$$

$$y'(5) = 0 \Rightarrow a_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}5} - a_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}5} = 0$$

$$\Rightarrow e^{5\sqrt{-\lambda}} a_1 - e^{-5\sqrt{-\lambda}} a_2 = 0$$

$$\Rightarrow e^{5\sqrt{-\lambda}} a_1 + e^{-5\sqrt{-\lambda}} a_1 = 0$$

$$\Rightarrow 2a_1 \cosh(5\sqrt{-\lambda}) = 0$$

$$\Rightarrow a_1 = 0$$

$$\Rightarrow a_2 = 0$$

$\Rightarrow$ trivial solution

Case 2) $\lambda = 0$

$$\Rightarrow y(x) = a_1 + a_2 x$$

$$y(0) = 0 \Rightarrow a_1 = 0$$

$$y'(x) = a_2$$

$$y'(5) = 0 \Rightarrow a_2 = 0$$

$\Rightarrow$ trivial solution

Case 3) $\lambda > 0$

$$\Rightarrow y(x) = a_1\cos(\sqrt{\lambda}x) + a_2\sin(\sqrt{\lambda}x)$$

$$y(0) = 0 \Rightarrow a_1 = 0$$

$$y'(x) = \sqrt{\lambda}\, a_2 \cos(\sqrt{\lambda}x)$$

$$y'(5) = 0 \Rightarrow \sqrt{\lambda}\, a_2 \cos(\sqrt{\lambda}\,5) = 0$$

$$\Rightarrow \cos(5\sqrt{\lambda}) = 0 \qquad (\text{for } a_2 \neq 0)$$

$$\Rightarrow \quad 5\sqrt{\lambda} = \left(n-\tfrac{1}{2}\right)\pi \quad , \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad \boxed{\lambda_n = \frac{\left(n-\frac{1}{2}\right)^2 \pi^2}{25}} \qquad \text{eigenvalues}$$

$$\Rightarrow \quad \boxed{\begin{aligned} y_n(x) &= \sin\left(\sqrt{\lambda_n}\, x\right) \\ &= \sin\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{5}\right) \end{aligned}} \qquad \text{eigenfunctions}$$

(2)

$$u_t = \alpha^2 u_{xx}$$

$$u_x(0,t) = 0$$

$$u_x(1,t) = 0$$

$$u(x,0) = f(x)$$

Assume $\quad u(x,t) = X(x)\, T(t)$

$$\Rightarrow \quad X T' = \alpha^2 X'' T$$

$$\Rightarrow \quad \frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda$$

convenience

$$X'' + \lambda X = 0$$

$$X'(0) = 0$$

$$X'(1) = 0$$

For nontrivial solns need $\lambda \geq 0$

$\boxed{\lambda = 0} \quad \Rightarrow \quad X(x) = a_1 + a_2 x$

$X'(0) = X'(1) = 0 \quad \Rightarrow \quad a_2 = 0$

$\Rightarrow \quad X_0(x) = 1$ is a soln with $\lambda_0 = 0$

$\boxed{\lambda > 0}$

$$\Rightarrow \quad X(x) = a_1 \cos(\sqrt{\lambda}\, x) + a_2 \sin(\sqrt{\lambda}\, x)$$

$$\Rightarrow \quad X'(x) = -a_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x) + a_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$

$$X'(0) = 0 \quad \Rightarrow \quad a_2 = 0$$

$$X'(1) = 0 \quad \Rightarrow \quad -a_1 \sqrt{\lambda} \sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \quad \sin(\sqrt{\lambda}) = 0 \qquad (\text{for } a_1 \neq 0)$$

$$\Rightarrow \quad \sqrt{\lambda} = n\pi \quad , \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad \boxed{\lambda_n = n^2\pi^2}$$

$$\Rightarrow \quad \boxed{X_n(x) = \cos(n\pi x)}$$

$$T_n' + \alpha^2 \lambda_n T_n = 0$$

$$\Rightarrow \quad T_n(t) = e^{-\alpha^2 \lambda_n t} \quad \text{(or any multiple of this)}$$

$$\Rightarrow \quad u_n(x,t) = X_n(x)\,T_n(t) \quad \text{is a soln for each } n = 1, 2, \ldots$$

By the principle of superposition,

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n u_n(x,t)$$

- the "$\frac{1}{2}$" is for convenience
- notice $u_0(x,t) = 1$

$$= \boxed{\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi x)\, e^{-\alpha^2 n^2 \pi^2 t}}$$

IC: $\quad f(x) = u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi x)$

which is a Fourier Cosine series

$$\Rightarrow \quad \boxed{c_n = 2\int_0^1 f(x)\cos(n\pi x) \quad , \quad n = 0, 1, 2, \ldots}$$

3 a) period of $\sec(3x) = \frac{1}{\cos(3x)}$
equals the period of $\cos(3x)$
which is $\boxed{\frac{2\pi}{3}}$

b) period of $\sin\left(\frac{x}{4}\right)$ is 8π
period of $\sin\left(\frac{x}{7}\right)$ is 14π
$\Rightarrow$ period of $\sin\left(\frac{x}{4}\right) + \sin\left(\frac{x}{7}\right)$

$$= \pi \times \text{lowest common multiple}(8, 14)$$

$$= \boxed{56\pi}$$

c) period of $\frac{1}{3 - \cos(x)}$
equals the period of $\cos(x)$
which is $\boxed{2\pi}$

d) period of $\ln\left(2 + \sin^2(x)\right)$
equals the period of $\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$
equals the period of $\cos(2x)$
which is $\boxed{\pi}$

④ $$f(x) = \begin{cases} 4 - 5x, & 0 < x < \frac{1}{2} \\ 5 - 5x, & \frac{1}{2} \leq x < 1 \end{cases}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)}$$

$$c_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx$$

$$= 2\int_0^{\frac{1}{2}} (4-5x) \sin(n\pi x)\, dx + 2\int_{\frac{1}{2}}^1 (5-5x) \sin(n\pi x)\, dx$$

$$= 8\int_0^{\frac{1}{2}} \sin(n\pi x)\, dx + 10\int_{\frac{1}{2}}^1 \sin(n\pi x)\, dx - 10\int_0^1 x \sin(n\pi x)\, dx$$

$$= -\frac{8}{n\pi} \cos(n\pi x)\Big|_0^{\frac{1}{2}} - \frac{10}{n\pi} \cos(n\pi x)\Big|_{\frac{1}{2}}^1$$

$$+ \frac{10}{n\pi} x \cos(n\pi x)\Big|_0^1 - \frac{10}{n\pi}\int_0^1 \cos(n\pi x)\, dx$$

$$= -\frac{8}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \frac{10}{n\pi}\left(\cancel{\cos(n\pi)} - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\cancel{+ \frac{10}{n\pi} \cos(n\pi)} - 0 \cancel{- \frac{10}{n^2\pi^2} \sin(n\pi x)\Big|_0^1}$$

$$= \frac{8}{n\pi} + \left(-\frac{8}{n\pi} + \frac{10}{n\pi}\right) \cos\left(\frac{n\pi}{2}\right)$$

$$= \frac{8}{n\pi} + \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right)$$

if n is even,

$$\cos\left(\frac{n\pi}{2}\right) = (-1)^{\frac{n}{2}}$$

if n is odd,

$$\cos\left(\frac{n\pi}{2}\right) = 0$$

n	$\cos\left(\frac{n\pi}{2}\right)$
1	0
2	−1
3	0
4	1
5	0
6	−1
7	0
8	1

thus $$\boxed{c_n = \begin{cases} \frac{8}{n\pi}, & n \text{ odd} \\ \frac{8}{n\pi} + \frac{2}{n\pi}(-1)^{\frac{n}{2}}, & n \text{ even} \end{cases}}$$