(1)

a) $f(x) = x^3 - x \quad , \quad L = 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 (x^3 - x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 x^3 \sin(n\pi x)\,dx - 2\int_0^1 x\sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi} x^3 \cos(n\pi x)\Big|_0^1 + \frac{6}{n\pi}\int_0^1 x^2\cos(n\pi x)\,dx - 2\int_0^1 x\sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \cancel{\frac{6}{n^2\pi^2} x^2 \sin(n\pi x)\Big|_0^1}$$

$$- \frac{12}{n^2\pi^2}\int_0^1 x\sin(n\pi x)\,dx - 2\int_0^1 x\sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi) - \left(\frac{12}{n^2\pi^2} + 2\right)\int_0^1 x\sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \left(\frac{12}{n^2\pi^2} + 2\right)\left[\frac{1}{n\pi} x\cos(n\pi x)\,dx\Big|_0^1 \cancel{- \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx}\right]$$

$$= \cancel{\frac{-2}{n\pi}\cos(n\pi)} + \left(\frac{12}{n^3\pi^3} + \cancel{\frac{2}{n\pi}}\right)\cos(n\pi)$$

$$= \frac{12}{n^3\pi^3}\cos(n\pi)$$

$$= \frac{12(-1)^n}{n^3\pi^3}$$

i.e. $$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3\pi^3}\sin(n\pi x)}$$

b) put $x = \frac{1}{2}$ into

$$x^3 - x = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3\pi^3} \sin(n\pi x)$$

$$\Rightarrow \frac{1}{8} - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3\pi^3} \sin\left(\frac{n\pi}{2}\right)$$

if n is even, $\sin\left(\frac{n\pi}{2}\right) = 0$

if n is odd, $\sin\left(\frac{n\pi}{2}\right) = (-1)^{\frac{n-1}{2}}$

$$\Rightarrow \frac{-3}{8} = \sum_{n=1,3,5,\dots}^{\infty} \frac{12(-1)}{n^3\pi^3} (-1)^{\frac{n-1}{2}}$$

$$\Rightarrow \boxed{\frac{\pi^3}{32} = \sum_{n=1,3,5,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots}$$

c) $\frac{\pi^3}{32} \approx 1 - \frac{1}{3^3}$

$$\Rightarrow \pi \approx \sqrt[3]{32\left(1 - \frac{1}{27}\right)} \approx \boxed{3.13511}$$

d) Parseval: $2\int_0^1 (x^3 - x)^2 \, dx = \sum_{n=1}^{\infty} b_n^2$

$$2\int_0^1 x^6 - 2x^4 + x^2 \, dx \qquad\qquad \sum_{n=1}^{\infty} \frac{144}{n^6\pi^6}$$

$$= 2\left(\frac{1}{7}x^7 - \frac{2}{5}x^5 + \frac{1}{3}x^3\right)\Big|_0^1$$

$$= 2\left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)$$

$$= 2\,\frac{15 - 42 + 35}{105}$$

$$= \frac{16}{105}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{144}\frac{16}{105} = \frac{\pi^6}{9 \cdot 105} = \boxed{\frac{\pi^6}{945}}$$

e) $\dfrac{\pi^6}{945} \approx 1 + \dfrac{1}{2^6}$

$$\Rightarrow \pi \approx \sqrt[6]{945\left(1+\frac{1}{64}\right)} \approx \boxed{3.14071}$$

f) abs. error in approx. found in (c) = 0.00648

(e) = 0.00088

thus approx found in $\boxed{(e)}$ is a better approx. to π.

(2)

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots$$

$$f(x-2h) = f(x) - 2hf'(x) + \underbrace{\frac{(2h)^2}{2}}_{= 2h^2} f''(x) - \underbrace{\frac{(2h)^3}{3!}}_{= \frac{4}{3}h^3} f'''(x) + \ldots$$

then $\frac{3}{2}f(x) - 2f(x-h) + \frac{1}{2}f(x-2h)$

$$= \cancel{\tfrac{3}{2}f(x)} - \cancel{2f(x)} + 2hf'(x) - h^2\cancel{f''(x)} + O(h^3)$$
$$+ \cancel{\tfrac{1}{2}f(x)} - hf'(x) + h^2\cancel{f''(x)} + O(h^3)$$

$$= hf'(x) + O(h^3)$$

$$\Rightarrow \boxed{\frac{\frac{3}{2}f(x) - 2f(x-h) + \frac{1}{2}f(x-2h)}{h} = f'(x) + O(h^2)}$$

as required

dx =	0.05	dt =	0.004	alpha^2	0.2	k=a dt/d)	0.32	pi	3.142		Offset	147	
Time	0.588	1	0.964	0.928	0.894	0.862	0.832	0.805	0.781	0.76	0.742	0.728	0.716
Time	**x**	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
0		0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
0.004		1	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
0.008		1	0.37	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
0.012		1	0.									0.5	0.55
0.016		1	0.									0.5	0.55
0.02		1	0.									0.5	0.55
0.024		1	0.									0.5	0.55
0.028		1	0.									0.5	0.55
0.032		1	0.									0.5	0.55
0.036		1	[illegible]									0.5	0.55
0.04		1	0.									0.5	0.55
0.044		1	0.									0.5	0.55
0.048		1	0.									.500	0.55
0.052		1	0.									.500	0.55
0.056		1	0.									.500	0.55
0.06		1	0.									.501	0.55
0.064		1	0.									.501	0.550
0.068		1	0.									.501	0.550
0.072		1	0.									.502	0.550
0.076		1	0.									.503	0.551
0.08		1	0.									.504	0.551
0.084		1	0.									.505	0.551
0.088		1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.506	0.552
0.092		1	0.843	0.699	0.580	0.492	0.437	0.412	0.413	0.433	0.466	0.507	0.552
0.096		1	0.847	0.707	0.590	0.502	0.447	0.420	0.419	0.437	0.469	0.508	0.553
0.1		1	0.851	0.714	0.599	0.512	0.456	0.428	0.426	0.442	0.471	0.510	0.553
0.104		1	0.855	0.721	0.608	0.522	0.465	0.436	0.432	0.446	0.474	0.511	0.554
0.108		1	0.859	0.728	0.617	0.531	0.474	0.444	0.438	0.450	0.477	0.513	0.555
0.112		1	0.862	0.734	0.625	0.540	0.483	0.452	0.444	0.455	0.48	0.515	0.556
0.116		1	0.865	0.740	0.633	0.549	0.491	0.459	0.450	0.459	0.483	0.517	0.556
0.12		1	0.868	0.746	0.640	0.557	0.500	0.466	0.456	0.464	0.486	0.519	0.557
0.124		1	0.871	0.751	0.648	0.565	0.507	0.474	0.462	0.469	0.490	0.521	0.559