MATH 257/316 Assignment 8
Due: Mon, Nov 23 in class

Problem 1 (hand in printed graph):
For this problem you will numerically solve the BVP of Problem 2 in Excel with:

$$\alpha^2 = 0.2\ , \qquad L = 1\ , \qquad q = 2\ , \qquad f(x) = 2 - 2x\ .$$

Download the spreadsheet *Heat_Assg8Prob1.xls*. This spreadsheet is setup to solve the problem but you will need to make some changes:

a) Change the left BC from $u(0,t) = 0$ to $u_x(0,t) = 0$.

b) Change the right BC from $u(1,t) = q$ to $u_x(1,t) = q$.

c) Fill the middle of the spreadsheet with an appropriate formula describing the PDE, $u_t = \alpha^2 u_{xx} - u$.

The given graph should show the temperature at times $t = 0, 0.2, 0.5, 1$ and 2. The curve corresponding to $t = 0$ will be the initial condition, $f(x) = 2 - 2x$. If the spreadsheet is completed correctly the curve corresponding to $t = 2$ should be a good approximation to the steady-state solution of the BVP.

d) Print out the graph and hand it in.

Problem 2 (do not hand in):
Consider a slender, homogeneous, conducting bar of uniform cross section that lies on the x-axis with ends at $x = 0$ and $x = L > 0$. The lateral surface of the bar radiates heat into the surroundings which is of ambient temperature $u = 0$. The end $x = 0$ is insulated and at the end $x = L$ there is a constant flux q. Assume the initial temperature distribution is a known function, $f(x)$.

$$\begin{aligned} & u_t = \alpha^2 u_{xx} - u, \qquad 0 < x < L,\ t > 0,\ \alpha > 0 \\ \text{BC}: \quad & u_x(0,t) = 0, \qquad u_x(L,t) = q \\ \text{IC}: \quad & u(x,0) = f(x) \end{aligned}$$

a) Find the steady-state solution of the above BVP.

b) Find the solution, $u(x,t)$, to the above BVP.

Problem 3 (do not hand in):
Consider a bar of length 1 and thermal diffusivity α^2 having a initial temperature distribution $\sin(\pi x)$. Suppose that the temperature at both end points is fixed at 0 and a heat source given by $\sin(3\pi x)$ is applied across the length of the bar. Find $u(x,t)$ that satisfies:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} + \sin(3\pi x), \qquad 0 < x < 1,\ t > 0,\ \alpha > 0 \\ \text{BC}: \quad & u(0,t) = 0, \qquad u(1,t) = 0 \\ \text{IC}: \quad & u(x,0) = \sin(\pi x) \end{aligned}$$

Problem 4 (do not hand in):
Consider the following boundary value problem that has a time-dependent, Dirichlet boundary condition:

$$\begin{aligned} & u_t = u_{xx}, && 0 < x < \pi,\ t > 0 \\ \text{BC}: \quad & u(0,t) = 0, && u(\pi,t) = t^2 \\ \text{IC}: \quad & u(x,0) = 0 && \end{aligned}$$

a) Solve the problem by the method of eigenfunction expansion.

b) Verify that

$$w(x,t) = \frac{1}{\pi}t^2 x + \frac{1}{3\pi}(x^2 - \pi^2)tx + \frac{1}{180\pi}(3x^4 - 10\pi^2 x^2 + 7\pi^4)x\ ,$$

satisfies the PDE (i.e. $w_t = w_{xx}$) and the boundary conditions.

c) Let $v(x,t) = u(x,t) - w(x,t)$ and solve by separation of variables. It may be useful to know that:

$$\int_0^\pi (3x^5 - 10\pi^2 x^3 + 7\pi^4 x)\sin(nx)\ dx = -\frac{360\pi(-1)^n}{n^5}\ .$$

d) Is your answer in (c) the same as your answer in (a)? Why or why not?