(2)

$$u_t = \alpha^2 u_{xx} - u$$

$$u_x(0,t) = 0$$

$$u_x(L,t) = q$$

$$u(0,t) = f(x)$$

a) The steady-state solution, $w(x)$ satisfies,

$$0 = \alpha^2 w''(x) - w(x)$$

$$\Rightarrow w''(x) = \frac{1}{\alpha^2} w(x)$$

$$\Rightarrow w(x) = A\cosh\left(\frac{x}{\alpha}\right) + B\sinh\left(\frac{x}{\alpha}\right)$$

for some constants A & B.

By matching the BCs we determine A & B:

$$w'(x) = \frac{A}{\alpha}\sinh\left(\frac{x}{\alpha}\right) + \frac{B}{\alpha}\cosh\left(\frac{x}{\alpha}\right)$$

$$w'(0) = 0 = \frac{B}{\alpha} \Rightarrow B = 0$$

$$w'(L) = q = \frac{A}{\alpha}\sinh\left(\frac{L}{\alpha}\right) \Rightarrow A = \frac{\alpha q}{\sinh\left(\frac{L}{\alpha}\right)}$$

thus $\boxed{w(x) = \alpha q \dfrac{\cosh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)}}$

b) Let $v(x,t) = u(x,t) - w(x)$ (the transient solution).

$$u_t = \alpha^2 u_{xx} - u$$

$$\Rightarrow (v+w)_t = \alpha^2 (v+w)_{xx} - (v+w)$$

$$\Rightarrow v_t = \alpha^2 v_{xx} - v + \cancel{\alpha^2 w_{xx} - w}$$

Also

$$v_x(0,t) = 0$$

$$v_x(L,t) = 0$$

$$v(0,t) = f(x) - w(x)$$

Solve by separation of variables:

write $v(x,t) = X(x)T(t)$

$$\Rightarrow XT' = \alpha^2 X''T - XT$$

$$\Rightarrow \frac{1}{\alpha^2}\left(\frac{T'}{T} + 1\right) = \frac{X''}{X} = k$$

We have
$$X'' - kX = 0$$
$$X'(0) = 0$$
$$X'(L) = 0$$

$k > 0 \Rightarrow$ trivial soln

$k = 0 \Rightarrow X(x) = Ax + B$

BCs $\Rightarrow A = 0$, take $X_0(x) = 1$

$k < 0$, write $k = -\lambda^2$

$$\Rightarrow X'' + \lambda^2 X = 0$$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$X'(0) = 0 \Rightarrow B = 0$

$X'(L) = 0 \Rightarrow 0 = -\lambda A\sin(\lambda L)$

$$\Rightarrow \lambda_n = \frac{n\pi}{L}$$

take $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

Then $T' + (\alpha^2\lambda^2 + 1)T = 0$

$$\Rightarrow T_n(t) = e^{-\left(\frac{\alpha^2 n^2\pi^2}{L^2} + 1\right)t}$$

$$\Rightarrow v(x,t) = \frac{c_0}{2}e^{-t} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\alpha^2 n^2\pi^2}{L^2} + 1\right)t}\cos\left(\frac{n\pi x}{L}\right)$$

Initial condition gives $\boxed{c_n = \frac{2}{L}\int_0^L (f(x) - w(x))\cos\left(\frac{n\pi x}{L}\right)dx}$, $\forall n$

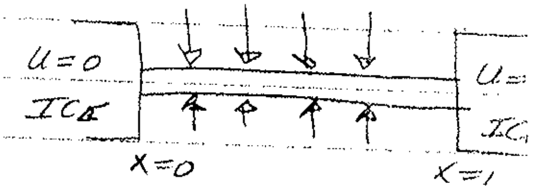
Finally, $\boxed{u(x,t) = \alpha q\frac{\cosh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)} + \frac{c_0}{2}e^{-t} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\alpha^2 n^2\pi^2}{L^2} + 1\right)t}\cos\left(\frac{n\pi x}{L}\right)}$

(3) : A BAR WITH AN EXTERNAL HEAT SOURCE $S(x)$

$u_t = \alpha^2 u_{xx} + \sin(3\pi x)$

BC: $u(0,t) = 0 = u(1,t)$

IC: $u(x,0) = \sin(\pi x)$



METHOD 1: STEADY SOLN $\frac{\partial u_\infty}{\partial t} = 0$ $\quad \alpha^2 u_\infty'' + \sin(3\pi x) = 0$

$\therefore \quad u_\infty' = + \frac{\cos(3\pi x)}{\alpha^2(3\pi)} + A \qquad u_\infty(x) = \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} + Ax + B$

$u_\infty(0) = B = 0 \qquad u_\infty(1) = \frac{\sin(3\pi)}{\alpha^2(3\pi)^2} + A = 0 \Rightarrow u_\infty(x) = \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}$

LET $u(x,t) = u_\infty(x) + v(x,t) \Rightarrow (u_\infty + v)_t = \alpha^2(u_\infty + v)_{xx} + \sin(3\pi x)$

$\therefore v_t = \alpha^2 v_{xx}$

$0 = u(0,t) = u_\infty(0) + v(0,t) \Rightarrow v(0,t) = 0$

$0 = u(1,t) = u_\infty(1) + v(1,t) \Rightarrow v(1,t) = 0$

$u(x,0) = u_\infty(x) + v(x,0) = \sin(\pi x) \Rightarrow v(x,0) = \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}$

BY SEPARATION OF VARIABLES

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2(n\pi)^2 t} \sin(n\pi x)$$

WHERE $v(x,0) = \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} = b_1 \sin(\pi x) + b_2 \sin(2\pi x) + b_3 \sin(3\pi x) + \cdots$

$\therefore \quad b_1 = 1 \qquad b_3 = -\frac{1}{\alpha^2(3\pi)^2} \qquad b_k = 0 \quad k \neq 1 \text{ or } 3$

$$v(x,t) = e^{-\alpha^2\pi^2 t}\sin(\pi x) - \frac{e^{-\alpha^2(3\pi)^2 t}}{\alpha^2(3\pi)^2}\sin(3\pi x)$$

$$u(x,t) = u_\infty(x) + v(x,t)$$

$$= \underbrace{\frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}}_{\text{STEADY}} + \underbrace{e^{-\alpha^2\pi^2 t}\sin(\pi x) - \frac{e^{-\alpha^2(3\pi)^2 t}}{\alpha^2(3\pi)^2}\sin(3\pi x)}_{\text{TRANSIENT}}$$

$$u(x,t) = e^{-\alpha^2\pi^2 t}\sin(\pi x) + \frac{\left(1 - e^{-\alpha^2(3\pi)^2 t}\right)}{\alpha^2(3\pi)^2}\sin(3\pi x)$$

METHOD 2: USING EIGENFUNCTION EXPANSIONS:

THE APPLICABLE EIGENVALUE PROBLEM IS:

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \quad X = A\cos\lambda x + B\sin\lambda x \quad X(0) = A = 0$$

$$X(1) = B\sin\lambda = 0 \quad \lambda_n = n\pi \quad n = 1, 2, \ldots \quad X_n = \sin(n\pi x)$$

LET $u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(t)\sin(n\pi x)$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{d\hat{u}_n}{dt}\sin(n\pi x) \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \hat{u}_n\left\{-(n\pi)^2\sin(n\pi x)\right\}$$

$$\therefore\; u_t - \alpha^2 u_{xx} = \sum_{n=1}^{\infty}\left\{\frac{d\hat{u}_n}{dt} + \alpha^2(n\pi)^2\hat{u}_n\right\}\sin(n\pi x) = \sin(3\pi x)$$

$$\therefore\; \frac{d\hat{u}_n}{dt} + \alpha^2(n\pi)^2\hat{u}_n = \left\{\begin{array}{ll} 1 & n = 3 \\ 0 & n \neq 3 \end{array}\right\} = \delta_{n3}$$

$$\frac{d}{dt}\left(e^{\alpha^2(n\pi)^2 t}\hat{u}_n\right) = e^{\alpha^2(n\pi)^2 t}\delta_{n3}$$

$$e^{\alpha^2(n\pi)^2 t}\hat{u}_n = \frac{e^{\alpha^2(n\pi)^2 t}}{\alpha^2(n\pi)^2}\delta_{n3} + C_n$$

$$\hat{u}_n(t) = \frac{\delta_{n3}}{\alpha^2(n\pi)^2} + C_n e^{-\alpha^2(n\pi)^2 t}$$

$$\therefore\; u(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\delta_{n3}}{\alpha^2(n\pi)^2} + C_n e^{-\alpha^2(n\pi)^2 t}\right\}\sin(n\pi x)$$

$$= \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} + \sum_{n=1}^{\infty} C_n e^{-\alpha^2(n\pi)^2 t}\sin(n\pi x)$$

NOW $u(x,0) = \sin(\pi x) = \dfrac{\sin(3\pi x)}{\alpha^2(3\pi)^2} + \sum_{n=1}^{\infty} C_n\sin(n\pi x)$

$$\therefore\; \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} = C_1\sin(\pi x) + C_2\sin(2\pi x) + C_3\sin(3\pi x) + \cdots$$

$C_1 = 1 \quad C_3 = -1/\alpha^2(3\pi)^2 \qquad C_k = 0 \quad k \neq 1, 3$

$$\therefore\; u(x,t) = e^{-\alpha^2(n\pi)^2 t} + \frac{\left(1 - e^{-\alpha^2(3\pi)^2 t}\right)}{\alpha^2(3\pi)^2}\sin(3\pi x)$$

(4) $u_t = u_{xx}$

$u(0,t) = 0$

$u(\pi,t) = t^2$

$u(x,0) = 0$

a) The function $w(x,t) = t^2 \frac{x}{\pi}$ satisfies the BCs.

Let $v(x,t) = u(x,t) - w(x,t)$

Then $u_t = u_{xx}$

$\Rightarrow (v+w)_t = (v+w)_{xx}$

$\Rightarrow v_t + \frac{2}{\pi} t x = v_{xx}$

$\Rightarrow v_t = v_{xx} - \frac{2}{\pi} t x$

Also $v(0,t) = 0$

$v(\pi,t) = 0$

$v(x,0) = u(x,0) - w(x,0) = 0$

We solve by the method of eigenfunction ~~expansion~~ expansion.

Here eigenfunctions are $\sin(nx)$, $n = 1, 2, 3, \ldots$

Write $v(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin(nx)$

$v_t = v_{xx} - \frac{2}{\pi} t x$

$$\Rightarrow \sum_{n=1}^{\infty} V_n'(t) \sin(nx) = \sum_{n=1}^{\infty} V_n(t)(-n^2)\sin(nx) + \sum_{n=1}^{\infty} b_n(t)\sin(nx)$$

where $b_n(t) = \frac{2}{\pi}\int_0^{\pi} -\frac{2}{\pi} t x \sin(nx)\, dx$

$= \frac{-4t}{\pi^2}\int_0^{\pi} x \sin(nx)\, dx$

$$= \frac{-4t}{\pi^2}\left(\frac{-1}{n} x \cos(nx) \Big|_0^{\pi} + \cancel{\frac{1}{n}\int_0^{\pi} \cos(nx)\, dx} \right)$$

$$= \frac{-4t}{\pi^2} \frac{-1}{n}\left(\pi \cos(n\pi) - 0 \right)$$

$$= \frac{4t}{n\pi} \cos(n\pi)$$

$$= \frac{4t(-1)^n}{n\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[V_n'(t) + n^2 V_n(t) - \frac{4t(-1)^n}{n\pi} \right] \sin(nx) = 0$$

$$\Rightarrow V_n'(t) + n^2 V_n(t) = \frac{4t(-1)^n}{n\pi}$$

$$\Rightarrow \frac{d}{dt}\left(e^{n^2 t} V_n(t) \right) = \frac{4t(-1)^n}{n\pi} e^{n^2 t} \qquad \text{(integrating factor method)}$$

$$\Rightarrow e^{n^2 t} V_n(t) = \frac{4(-1)^n}{n\pi} \int t e^{n^2 t}\, dt + C_n$$

$$= \frac{1}{n^2} t e^{n^2 t} - \frac{1}{n^2} \int e^{n^2 t}\, dt$$

$$= \frac{1}{n^2} t e^{n^2 t} - \frac{1}{n^4} e^{n^2 t}$$

$$= \frac{1}{n^2} e^{n^2 t}\left(t - \frac{1}{n^2} \right)$$

$$\Rightarrow V_n(t) = \frac{4(-1)^n}{n\pi} \frac{1}{n^2}\left(t - \frac{1}{n^2} \right) + C_n e^{-n^2 t}$$

$$IC: \quad 0 = v(x,0) = \sum_{n=1}^{\infty} V_n(0)\sin(nx)$$

$$\Rightarrow V_n(0) = 0, \quad \forall n$$

Here $V_n(0) = \frac{4(-1)^n}{n\pi}\frac{1}{n^2}\left(\frac{-1}{n^2}\right) + c_n$

$$\Rightarrow c_n = \frac{4(-1)^n}{\pi n^5}$$

Thus $u(x,t) = w(x,t) + v(x,t)$

$$\boxed{u(x,t) = \frac{1}{\pi}t^2 x + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi n^3}\left(t - \frac{1}{n^2}\right) + \frac{4(-1)^n}{\pi n^5}e^{-n^2 t}\right)\sin(nx)}$$

b) $w(x,t) = \frac{1}{\pi}t^2 x + \frac{1}{3\pi}(x^2 - \pi^2)tx$

$$+ \frac{1}{180\pi}\left(3x^4 - 10\pi^2 x^2 + 7\pi^4\right)x$$

$w(0,t) = 0 + 0 + 0 = 0$ ✓

$w(\pi,t) = t^2 + 0 + \frac{1}{180\pi}\left(3\pi^4 - 10\pi^4 + 7\pi^4\right)\pi = t^2$ ✓

$w_t(x,t) = \frac{2}{\pi}tx + \frac{1}{3\pi}(x^2 - \pi^2)x$

$$w_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\left(\frac{t}{3\pi}x^3 + \frac{1}{60\pi}x^5 - \frac{\pi}{18}x^3\right)$$

$$= \frac{2t}{\pi}x + \frac{1}{3\pi}x^3 - \frac{\pi}{3}x$$

$$= \frac{2}{\pi}tx + \frac{1}{3\pi}(x^2 - \pi^2)x$$

$$= w_t(x,t)$$ ✓

c) Let $v(x,t) = u(x,t) - w(x,t)$

$$\Rightarrow \quad v_t = v_{xx}$$

$$v(0,t) = 0$$

$$v(\pi,t) = 0$$

$$v(x,0) = u(x,0) - w(x,0)$$

$$= \frac{-1}{180\pi}\left(3x^4 - 10\pi^2 x^2 + 7\pi^4\right)x$$

Separation of variables:

$$v(x,t) = X(x)\,T(t)$$

$$v_t = v_{xx}$$

$$\Rightarrow \quad XT' = X''T$$

$$\Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda^2 \quad , \text{ for nontrivial solutions}$$

We have,

$$X'' + \lambda^2 X = 0$$

$$X(0) = 0$$

$$X(\pi) = 0$$

Then $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$

$$X(0) = 0 = A$$

$$X(\pi) = 0 = B\sin(\lambda\pi)$$

$$\Rightarrow \quad \lambda_n = n \quad \text{— eigenvalues}, \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad X_n(x) = \sin(n\pi) \quad \text{— eigenfunctions}$$

Also,

$$T' + \lambda^2 T = 0$$

$$\Rightarrow \quad T_n(t) = e^{-n^2 t}$$

Then $$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$$

$$v(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

$$\Rightarrow \quad c_n = \frac{2}{\pi}\int_0^{\pi} v(x,0)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{-1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x\sin(nx)\,dx$$

~~$= \frac{-1}{90\pi^2}\left[3\int_0^{\pi} x^5\sin(nx)\,dx - 10\pi^2\int_0^{\pi} x^3\sin(nx)\,dx + 7\pi^4\int_0^{\pi} x\sin(nx)\,dx\right]$~~

$$= \frac{-1}{90\pi^2}\int_0^{\pi}\left(3x^5 - 10\pi^2x^3 + 7\pi^4x\right)\sin(nx)\,dx$$

$$= \frac{-1}{90\pi^2}\left(\frac{-360\pi(-1)^n}{n^5}\right)$$

$$= \frac{4(-1)^n}{\pi n^5}$$

Thus $$u(x,t) = w(x,t) + v(x,t)$$

$$\boxed{u(x,t) = \frac{1}{\pi}t^2x + \frac{1}{3\pi}(x^2-\pi^2)tx + \frac{1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi n^5}e^{-n^2t}\sin(nx)}$$

d) The answers appear different but are the same because $\sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi n^3}\left(t - \frac{1}{n^2}\right)$ is the Fourier sine series of

$$\frac{1}{3\pi}(x^2-\pi^2)tx + \frac{1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x.$$