MATH 257/316 ASSIGNMENT 1

PROBLEM 1: $u(x,t) = t^{-1/2} f(s) \qquad s = x t^{-1/2}$

$$u_t = -\frac{1}{2} t^{-3/2} f(s) + t^{-1/2} f'(s) . x\left(-\frac{1}{2} t^{-3/2}\right)$$

$$= -\frac{1}{2} t^{-3/2} \left\{ f(s) + (x t^{-1/2}) f'(s) \right\}$$

$$= -\frac{1}{2} t^{-3/2} \left(f(s) + s f'(s) \right)$$

$$u_x = t^{-1/2} f'(s) . t^{-1/2} = t^{-1} f'(s)$$

$$u_{xx} = t^{-1} f''(s) . t^{-1/2} = t^{-3/2} f''(s)$$

NOW SUBSTITUTE THESE EXPRESSIONS INTO THE HEAT EQ

$$0 = u_t - u_{xx}$$

$$= -\frac{1}{2} t^{-3/2} (f(s) + s f'(s)) - t^{-3/2} f''(s)$$

THUS $\quad f''(s) + \frac{1}{2}(s f'(s) + f(s)) = 0$

NOW USING THE HINT NOTICE THAT $\quad s f'(s) + f(s) = (s f(s))'$

$$\therefore \quad f''(s) + \frac{1}{2}(s f(s))' = 0$$

$$\Rightarrow \quad f'(s) + \frac{1}{2} s f(s) = A$$

THIS IS JUST A LINEAR 1ST ORDER ODE WITH INTEGRATING FACTOR $F = e^{s^2/4}$

$$\therefore \quad \left(e^{s^2/4} f(s)\right)' = A e^{s^2/4}$$

$$\Rightarrow \quad e^{s^2/4} f(s) = A \int^s e^{x^2/4} dx + B$$

$$f(s) = e^{-s^2/4}\left[A \int^s e^{x^2/4} dx + B\right]$$

NOW SINCE $f(s)$ SHOULD DECAY AT $\pm\infty$ ASSUME $A = 0$ SO THAT

$$f(s) = B e^{-s^2/4}$$

THUS $\quad u(x,t) = B t^{-1/2} e^{-x^2/4t}$

NOW WE ARE GIVEN THAT $\quad 1 = \int_{-\infty}^{\infty} u \, dx$

$$= B t^{-1/2} \int_{-\infty}^{\infty} e^{-(x^2/4t)} dx$$

LET $\xi = x/2\sqrt{t} \Rightarrow d\xi = dx/2\sqrt{t}$ THUS

$$1 = 2B \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 2B\sqrt{\pi} \Rightarrow B = \frac{1}{2\sqrt{\pi}}$$

THUS $\quad u(x,t) = \dfrac{e^{-x^2/4t}}{2\sqrt{\pi t}}$

PROBLEM 2: $u(x,t) = f(x+ct) + g(x-ct)$

$$u_t = c f'(x+ct) - c g'(x-ct)$$
$$u_{tt} = c^2 f''(x+ct) + c^2 g'(x-ct)$$
$$u_x = f'(x+ct) + g'(x-ct)$$
$$u_{xx} = f''(x+ct) + g''(x-ct)$$
$$u_{tt} - c^2 u_{xx} = \{c^2 f'' + c^2 g''\} - c^2\{f'' + g''\} = 0$$

PROBLEM 3:

(a) $2y'' - 5y' - 3y = 0$ LET $y = e^{rx} \Rightarrow (2r^2 - 5r - 3)e^{rx} = 0$ $r = -\frac{1}{2}$ $r = 3$

$\therefore y(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$

(b) $y^{(4)} + 2y'' + y = 0$; $y = e^{rx} \Rightarrow (r^2+1)^2 e^{rx} = 0$ $r = \pm i$ ARE DOUBLE ROOTS

$y = A_1 e^{ix} + A_2 e^{-ix} + A_3 x e^{ix} + A_4 x e^{-ix}$

OR $y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$

(c) $Ly = y'' - 5y' + 4y = 8e^x$

LOOK FOR A SOLUTION y_H TO THE HOMOGENEOUS EQ $Ly_H = 0$

LET $y_H = e^{rx} \Rightarrow (r^2 - 5r + 4)e^{rx} = 0$ $r = 1, 4$ ARE ROOTS, SO $y_H = C_1 e^x + C_2 e^{4x}$

WE NOW NEED TO LOOK FOR A PARTICULAR SOLUTION $y_p(x)$. SINCE THE FORCING TERM $8e^x$ IS A CONSTANT MULTIPLE OF ONE SOLUTIONS TO THE HOMOGENEOUS EQ WE ASSUME $y_p(x) = Axe^x \Rightarrow y_p' = A(xe^x + e^x)$; $y_p'' = A(xe^x + 2e^x)$

$\therefore Ly_p = A(xe^x + 2e^x) - 5A(xe^x + e^x) + 4Axe^x$

$= Ax\{e^x - 5e^x + 4e^x\} + A(2-5)e^x = 8e^x \Rightarrow A = -8/3$

$\therefore y_p(x) = C_1 e^x + C_2 e^{4x} - 8/3\, x e^x$

(d) $Ly = x^2 y'' - 2xy' - 4y = 0$ C-E EQ $\Rightarrow y = x^r$ $y' = rx^{r-1}$ $y'' = r(r-1)x^{r-2}$

$\therefore Ly = [r(r-1) - 2r - 4]x^r = (r^2 - 3r - 4)x^r = 0 \Rightarrow r = -1, 4$

$y = C_1 x^{-1} + C_2 x^4$

(e) $Ly = 4x^2 y'' + 8xy' + y = 0$, $y = x^r \Rightarrow Ly = [4r(r-1) + 8r + 1]x^r = (4r^2 + 4r + 1)x^r = 0$

$\therefore r = -1/2$ IS A DOUBLE ROOT $\Rightarrow$ ONE SOL'N IS $y(x) = x^{-1/2}$

TO OBTAIN A 2ND SOLUTION $\frac{\partial}{\partial r} y(x,r)\Big|_{r=-1/2} = x^r \ln x \Big|_{r=-1/2} = x^{-1/2} \ln x$

$\therefore y(x) = C_1 x^{-1/2} + C_2 x^{-1/2} \ln x$

3(f) $Ly = x^2y'' + 3xy' + 3y = 0 \quad y(1)=1 \quad y'(1) = -5$

$C\text{-}E \Rightarrow y = x^r;\ Ly = [r(r-1) + 3r + 3]x^r = [r^2 + 2r + 3]x^r = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4\cdot 3}}{2} = -1 \pm \sqrt{2}\, i$$

$\therefore\ y(x) = x^{-1}[A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x)]$

$\boxed{y(1) = A = 1}$ $\quad y' = -x^{-2}[A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x)]$

$\quad + x^{-1}\left[-\frac{A\sqrt{2}}{x}\sin(\sqrt{2}\ln x) + \frac{B\sqrt{2}}{x}\cos(\sqrt{2}\ln x)\right]$

$-5 = y'(1) = -A + [0 + B\sqrt{2}] \Rightarrow B = -4/\sqrt{2} = -2\sqrt{2}$.

$\therefore\ y(x) = x^{-1}[\cos(\sqrt{2}\ln x) - 2\sqrt{2}x^{-1}\sin(\sqrt{2}\ln x)$

PROBLEM 4: $(1-x)y' - y = 0$

1. $y' - \frac{1}{1-x}y = 0 \quad F = e^{-\int \frac{dx}{1-x}} = e^{\ln(1-x)} = (1-x)$

$\therefore\ [(1-x)y]' = 0 \Rightarrow \boxed{y = \frac{C}{1-x}}$

2. $y = C(1 + x + x^2 + \cdots)$ THIS IS JUST A GEOMETRIC SERIES WHICH CONVERGES FOR ALL $|x| < 1$

3. LET $y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$Ly = (1-x)y' - y = \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$.

$\therefore\ \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n(n+1)x^n$

$m = n-1 \Rightarrow n = m+1 \qquad n = m \quad \Rightarrow \sum_{m=0}^{\infty}[a_{m+1}(m+1) - a_m(m+1)]x^m = 0$

$n=1 \Rightarrow m=0$

SINCE X IS ARBITRARY WE MUST HAVE THAT $a_{m+1} = a_m \quad m = 0, 1, \ldots$

$\therefore\ a_0 = a_1 = a_2 = \cdots$

$\Rightarrow\ y(x) = a_0[1 + x + x^2 + \cdots] = \frac{a_0}{1-x}$ IF $|x| < 1$

- THE SERIES IS THE SAME AS THAT IN 4.2
- THE COEFFICIENT $1-x$ OF y' VANISHES ~~AT~~ $X=1$ AT WHICH POINT THE SERIES DIVERGES. $X=1$ IS A SINGULAR POINT OF THE ODE.