1

$$(x-1)y'' - (x-3)y' - y = 0 \quad , \quad \begin{array}{l} y(0) = 3 \\ y'(0) = 3 \end{array}$$

$x_0 = 0$

write $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow (x-1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - (x-3)\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} -n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty}(n(n-1) + 3n)a_n x^{n-1} + \sum_{n=0}^{\infty}(-n-1)a_n x^n = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} -(m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty}((m+1)m + 3(m+1))a_{m+1}x^m + \sum_{m=0}^{\infty}(-m-1)a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \underbrace{\left(-(m+2)(m+1)a_{m+2} + (m+1)(m+3)a_{m+1} - (m+1)a_m\right)}_{=0 \ \forall m \geq 0} x^m = 0 \quad \forall x$$

$$\Rightarrow a_{m+2} = \frac{m+3}{m+2}a_{m+1} - \frac{1}{m+2}a_m$$

ICs: $a_0 = 3$

$a_1 = 3$

$$\Rightarrow a_2 = \frac{3}{2}a_1 - \frac{1}{2}a_0 = \frac{9}{2} - \frac{3}{2} = 3$$

notice whenever $a_{m+1} = a_m$

$$a_{m+2} = \frac{m+3}{m+2}a_m - \frac{1}{m+2}a_m = \frac{m+2}{m+2}a_m = a_m$$

$\Rightarrow a_m = 3 \ , \ \forall m \geq 0$

$$\Rightarrow \boxed{y(x) = 3\sum_{n=0}^{\infty} x^n = \frac{3}{1-x}}$$

(2) $(x^2+1)y'' + \frac{7}{2}xy' + y = 0$

$x_0 = 0$

write $y = \sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow (x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \frac{7}{2}x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{(m=n-2)} + \underbrace{\sum_{n=0}^{\infty}\left(n(n-1) + \frac{7}{2}n + 1\right)a_n x^n}_{(m=n)} = 0$

$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty}\left(m^2 + \frac{5}{2}m + 1\right)a_m x^m = 0$

$\Rightarrow \sum_{m=0}^{\infty} \underbrace{\left((m+2)(m+1)a_{m+2} + (m+2)(m+\tfrac{1}{2})a_m\right)}_{=0 \quad \forall m \geq 0} x^m = 0 \quad \forall x$

$\Longrightarrow \quad a_{m+2} = \frac{-(m+\frac{1}{2})}{m+1}a_m = \frac{-(2m+1)}{2m+2}a_m$

$\Rightarrow \quad a_2 = -\frac{1}{2}a_0 \qquad a_3 = -\frac{3}{4}a_1$

$a_4 = \left(-\frac{5}{6}\right)\left(-\frac{1}{2}\right)a_0 \qquad a_5 = \left(-\frac{7}{8}\right)\left(-\frac{3}{4}\right)a_1$

$a_6 = \left(-\frac{9}{10}\right)\left(-\frac{5}{6}\right)\left(-\frac{1}{2}\right)a_0 \qquad a_7 = \left(-\frac{11}{12}\right)\left(-\frac{7}{8}\right)\left(-\frac{3}{4}\right)a_1$

$\vdots \qquad \vdots$

ie general solution is :

$$\boxed{\begin{aligned} y(x) &= a_0\left(1 - \frac{1}{2}x^2 + \frac{5\cdot 1}{6\cdot 2}x^4 - \frac{9\cdot 5\cdot 1}{10\cdot 6\cdot 2}x^6 + \dots\right) \\ &\quad + a_1\left(x - \frac{3}{4}x^3 + \frac{7\cdot 3}{8\cdot 4}x^5 - \frac{11\cdot 7\cdot 3}{12\cdot 8\cdot 4}x^7 + \dots\right)\end{aligned}}$$

3

$$y'' + x^2 y' + y = 0$$

$$x_0 = 0$$

write $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$(m = n-2)$ $(m = n+1)$ $(m = n)$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=1}^{\infty} (m-1) a_{m-1} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

(take out $m=0$ term)

$$(2 \cdot 1 \cdot a_2 + a_0) + \sum_{m=1}^{\infty} \underbrace{\left((m+2)(m+1) a_{m+2} + (m-1) a_{m-1} + a_m\right)}_{=0} x^m = 0$$

$$\boxed{a_2 = -\frac{1}{2} a_0}$$

$$a_{m+2} = \frac{-1}{(m+2)(m+1)} a_m - \frac{(m-1)}{(m+2)(m+1)} a_{m-1}, \quad m \geq 1$$

$$\Rightarrow \boxed{a_3 = \frac{-1}{6} a_1}$$

$$\Rightarrow a_4 = \frac{-1}{12} a_2 - \frac{1}{12} a_1$$

$$\boxed{a_4 = \frac{1}{24} a_0 - \frac{1}{12} a_1}$$

& $a_5 = \frac{-1}{20} a_3 - \frac{1}{10} a_2$

$$\boxed{a_5 = \frac{1}{120} a_1 + \frac{1}{20} a_0}$$

4

a)

$$p(x) = \frac{3}{x^2} \implies x = 0 \quad \text{singular point}$$

$$\lim_{x \to 0} x\,p(x) = \lim_{x \to 0} \frac{3}{x} \quad \text{DNE} \longrightarrow \boxed{x = 0 \ \text{irregular singular point}}$$

b)

$$p(x) = \frac{x+1}{(x^2-1)^2} = \frac{x+1}{(x+1)^2(x-1)^2} = \frac{1}{(x+1)(x-1)^2}$$

$$\implies x = -1,\ 1 \quad \text{singular points}$$

$$\lim_{x \to -1} (x+1)\,p(x) = \lim_{x \to -1} \frac{1}{(x-1)^2} = \frac{1}{4}, \quad \text{finite}$$

$$\lim_{x \to -1} (x+1)^2 q(x) = \lim_{x \to -1} (x+1)^2 \frac{x+2}{(x^2-1)^2}$$

$$= \lim_{x \to -1} \frac{x+2}{(x-1)^2} = \frac{1}{4} \quad \text{finite}$$

$$\implies \boxed{x = -1 \ \text{regular singular point}}$$

$$\lim_{x \to 1} (x-1)\,p(x) = \lim_{x \to 1} \frac{1}{(x+1)(x-1)} \quad \text{DNE}$$

$$\implies \boxed{x = 1 \ \text{irregular singular point}}$$

c) $q(x) = \dfrac{2}{\sin^2(x)}$ $\Rightarrow$ x singular whenever

$\sin(x) = 0$

$\Rightarrow x = n\pi$, $n \in \mathbb{Z}$ singular

$$\lim_{x \to n\pi} (x-n\pi)^2 q(x) = \lim_{x \to n\pi} \frac{2(x-n\pi)^2}{\sin^2(x)}$$

L'Hopital's rule

$$= \lim_{x \to n\pi} \frac{4(x-n\pi)}{2\sin(x)\cos(x)}$$

$$= \lim_{x \to n\pi} \frac{4(x-n\pi)}{\sin(2x)}$$

L'Hopital's rule

$$= \lim_{x \to n\pi} \frac{4}{2\cos(2x)}$$

$$= \frac{4}{2\cos(2n\pi)}$$

$$= 2 \text{ , finite , } \forall n$$

$$\lim_{x \to n\pi} (x-n\pi)\, p(x) = \lim_{x \to n\pi} \frac{(x-n\pi)\sin\left(\frac{x}{2}\right)}{\sin^2(x)}$$

L'Hopital's rule

$$= \lim_{x \to n\pi} \frac{\sin\left(\frac{x}{2}\right) + (x-n\pi)\frac{1}{2}\cos\left(\frac{x}{2}\right)}{\sin(2x)}$$

1) if n is odd $\sin\left(\frac{x}{2}\right) \to \sin\left(\frac{n\pi}{2}\right)$

$= 1$ or -1

$\Rightarrow$ limit DNE $\left(\frac{\text{const}}{0} \text{ case}\right)$

2) if n is even $\left(\frac{0}{0} \text{ case}\right)$

$$= \lim_{x \to n\pi} \frac{\cos\left(\frac{x}{2}\right) - (x-n\pi)\frac{1}{4}\sin\left(\frac{x}{2}\right)}{2\cos(2x)}$$

by L'Hopital's rule

$$= \frac{\cos\left(\frac{n\pi}{2}\right)}{2\cos(2n\pi)} \text{ is finite}$$

The functions have Taylor series

Thus $\boxed{\begin{array}{l} x = n\pi \text{ regular singular point if } n \text{ is even} \\ \text{irregular singular point if } n \text{ is odd} \end{array}}$

5 a) $p(x) = \frac{1}{x^2-3}$ has singularities at
$x = \pm\sqrt{3}$

$\Rightarrow$ radius of convergence of Taylor series of $p(x)$ centred at $x_0 = 0$ is
$|0 - \sqrt{3}| = \sqrt{3}$

$q(x) = \frac{\tan(x)}{x^2-3}$ has singularities at
$x = \pm\sqrt{3}$, $(n+\frac{1}{2})\pi$, $\forall n \in \mathbb{Z}$

$\Rightarrow$ radius of convergence of Taylor series of $q(x)$ centred at $x_0 = 0$ is
$\left|0 - \frac{\pi}{2}\right| = \frac{\pi}{2}$ (since $\frac{\pi}{2}$ (or $-\frac{\pi}{2}$) is the closest singularity to $x_0 = 0$)

Thus radius of convergence of

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \geq \min\left(\sqrt{3}, \frac{\pi}{2}\right) = \boxed{\frac{\pi}{2}}$$

b) $p(x) = \frac{x^3}{4x^2+9}$ has singularities at
$x = \pm\sqrt{-\frac{9}{4}} = \pm\frac{3i}{2}$

and same for $q(x)$

$\Rightarrow$ radius of convergence of $p(x)$ & $q(x)$ centred at $x_0 = 2$ is

$$\left|2 - \frac{3i}{2}\right| = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$\Rightarrow$ radius of convergence of series representation of $y(x)$ centred at $x_0 = 2$ is at least $\boxed{\frac{5}{2}}$