Chapter 25

Lecture 29 More Wedge Problems

Example 25.1 A wedge with Inhomogeneous BC

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 0 < r < a, \quad 0 < \theta < \alpha \tag{25.1}$$

$$u(r,0) = u_0 \quad u(r,\alpha) = u_1 \quad u(r,\theta) < \infty \text{ as } r \to 0 \quad u(a,\theta) = f(\theta). \tag{25.2}$$

Let us look for the simplest function of θ only that satisfies the inhomogeneous BC of the from: $w(\theta) = (u_1 - u_0)\frac{\theta}{\alpha} + u_0$. Note that $w_{\theta\theta} = 0$ and that $w(0) = u_0$ and $w(\alpha) = u_1$. Then let $u(r,\theta) = w(\theta) + v(r,\theta)$.

$$\left.\begin{array}{c} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 \\ v(r,0) = 0 \quad v(r,\alpha) = 0 \\ v(a,\theta) = f(\theta) - w(\theta) \end{array}\right\} \begin{array}{l}\text{Essentially the problem} \\ \text{solved in Example 24.2}\end{array} \tag{25.3}$$

The solution is

$$u(r,\theta) = (u_1 - u_0)\frac{\theta}{\alpha} + u_0 + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \sin\left(\frac{n\pi\theta}{\alpha}\right) \tag{25.4}$$

where

$$c_n = \frac{2}{\alpha} a^{-\left(\frac{n\pi}{\alpha}\right)} \int_0^\infty \left[f(\theta) - w(\theta)\right] \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta. \tag{25.5}$$

Example 25.2 A wedge with insulating BC on $\theta = 0$ and $\theta = \alpha < 2\pi$.

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ &u_\theta(r,0) = 0 \quad u_\theta(r,\alpha) = 0 \\ &u(a,\theta) = f(\theta). \end{aligned} \tag{25.6}$$

Let

$$u(r,\theta) = R(r)\Theta(\theta) \Rightarrow r^2\left(R'' + \frac{1}{r}R'\right)/R(r) = -\Theta''/\Theta = \lambda^2 \tag{25.7}$$

Θ equation⟩

$$\left.\begin{aligned} &\Theta'' + \lambda^2\Theta = 0 \\ &\Theta'(0) = 0 = \Theta'(\alpha) \end{aligned}\right\} \begin{aligned} &\Theta(\theta) = A\cos\lambda\theta + B\sin(\lambda\theta) \\ &\Theta'(0) = B\lambda = 0 \ \lambda = 0 \text{ or } B = 0; \end{aligned} \tag{25.8}$$

$$\begin{aligned} &\Theta'(\theta) = -A\lambda\sin(\lambda\theta) + B\lambda\cos(\lambda\theta) \\ &\Theta'(\alpha) = -A\lambda\sin(\lambda\alpha) = 0 \ \lambda_n = \tfrac{n\pi}{\alpha};\ n = 0, 1, \ldots \end{aligned} \tag{25.9}$$

R equation⟩ $r^2R_n'' + rR_n' - \lambda_n^2 R_n = 0$.

n = 0: $rR_0'' + R_0' = (rR_0')' = 0 \Rightarrow rR_0' = a?_0 \Rightarrow R_0(r) = a?_0 \ln r + c_0$.

n ≥ 1: $r^2R_n'' + rR_n' - \lambda^2 R_n = 0 \Rightarrow R_n = c_n r^{\lambda_n} + D_n r^{-\lambda_n}$.

Since $u(r,\theta) < \infty$ (i.e. must be bounded) as $r \to 0$ we require $d_0 = 0 =$

D_n. Therefore

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right) \tag{25.10}$$

$$f(\theta) = u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n a^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right) \tag{25.11}$$

$$c_0 = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)d?\theta \quad c_n = \frac{2}{\alpha}a^{-\left(\frac{n\pi}{\alpha}\right)}\int_0^{\alpha} f(\theta)\cos\left(\frac{n\pi\theta}{\alpha}\right)d\theta \tag{25.12}$$

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n r^{\left(\frac{n\pi}{\alpha}\right)} \cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{25.13}$$

Example 25.3 Mixed BC - a 'crack like' problem.

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{25.14}$$

subject to

$$u(r,0) = 0 \quad \frac{\partial u}{\partial \theta}(r,\pi) = 0 \tag{25.15}$$

$$u(a,\theta) = f(\theta). \tag{25.16}$$

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$r^2\frac{\left(R'' + \frac{1}{r}R'\right)}{R} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \tag{25.17}$$

Θ equation⟩

$$\begin{array}{ll} \Theta'' + \lambda^2\Theta = 0 & \Theta = A\cos\lambda\theta + B\sin\lambda\theta \quad \Theta' = -A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta \\ \Theta(0) = 0 \quad \Theta'(\pi) = 0 & \Theta(0) = A = 0 \quad \Theta'(\pi) = B\lambda\cos(\lambda\pi) = 0 \Rightarrow \pi\lambda_1 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \end{array} \tag{25.18}$$

or $\lambda_n = (2n+1)\frac{1}{2}$ $n = 0, 1, \ldots$ $\lambda \neq 0$ as this would be trivial.

R equation⟩ $r^2R'' + rR' - \lambda^2 R = 0$ $R(r) = r^\gamma \Rightarrow \gamma^2 - \lambda^2 = 0$ $\gamma = \pm\lambda$. Therefore

$$u_n(r,\theta) = \left(c_n r^{\lambda_n} + d_n r^{-\lambda_n}\right)\sin\lambda_n\theta. \tag{25.19}$$

Since u should be bounded as $r \to 0$ we conclude that $d_n = 0$. The general solution is thus

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^{(2n+1)/2}\sin\left(\frac{(2n+1)}{2}\theta\right) \tag{25.20}$$

$$f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} c_n a^{(2n+1)/2}\sin\left(\left(\frac{2n+1}{2}\right)\theta\right). \tag{25.21}$$

Check orthogonality

$$\int_0^\pi \sin\left(\left(\frac{2m+1}{2}\right)\theta\right)\sin\left(\left(\frac{2n+1}{2}\right)\theta\right)\,d\theta = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \end{cases}. \tag{25.22}$$

Therefore

$$c_n = \frac{2a^{-\left(n+\frac{1}{2}\right)}}{\pi}\int_0^\pi f(\theta)\sin\left(\left(n+\frac{1}{2}\right)\theta\right)\,d\theta \tag{25.23}$$

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^{\left(n+\frac{1}{2}\right)}\sin\left(\left(n+\frac{1}{2}\right)\theta\right). \tag{25.24}$$