HEAT EQUATION EXAMPLES

1. Find the solution to the heat conduction problem:

$$\begin{aligned} 4u_t &= u_{xx},\ 0 \le x \le 2,\ t > 0 \\ u(0,t) &= 0 \\ u(2,t) &= 0 \\ u(x,0) &= 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x) = f(x) \end{aligned}$$

Solution:

We use separation of variables. Let $u(x,t) = X(x)T(t)$. Then $4u_t = u_{xx}$ becomes $4X(x)T'(t) = X''(x)T(t)$. We divide both sides by $X(x)T(t)$ to obtain:

$$4\frac{T'}{T} = \frac{X''}{X} = \lambda, \tag{1}$$

where λ is a constant.

What happens to the boundary conditions under the separation of variables?

$$0 = u(0,t) = X(0)T(t) \Rightarrow X(0) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

$$0 = u(2,t) = X(2)T(t) \Rightarrow X(2) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

So we have $X(0) = X(2) = 0$. Can the initial condition tell us anything at this stage?

$$f(x) = u(x,0) = X(x)T(t) \Rightarrow T(t) = f(x)/X(x)???$$

No, it can't. The trick worked on the boundary conditions b/c they were homogeneous ($= 0$). We'll actually use the initial condition at the end to solve for constants.

Let's start with the T-equation from (3):

$$T'(t) = \frac{\lambda}{4}T(t).$$

Solving, we notice that this is a separable equation

$$\frac{dT}{dt} = \frac{\lambda}{4}T \Rightarrow \frac{dT}{T} = \frac{\lambda}{4}dt.$$

Integrating both sides,

$$\int \frac{dT}{T} = \int \frac{\lambda}{4}dt \Rightarrow \ln(T) = \frac{\lambda}{4}t + C \Rightarrow T(t) = Ce^{\lambda t/4},$$

taking the exponential of both sides.

Next we deal with the X-equation in (3) with conditions $X(0) = X(2) = 0$ derived from the boundary conditions

$$\begin{aligned} X'' &= \lambda X \\ X(0) &= X(2) = 0. \end{aligned}$$

This is an *eigenvalue problem*. There are 3 cases to consider: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

We begin with the $\lambda > 0$ case - note that we expect this to only yield the trivial solution (aka $X = 0$), since $T(t) = Ce^{\lambda t/4}$so $u(x,t) = X(x)T(t) = X(x)e^{\lambda t/4}$ and $\lambda > 0$ would suggest that the temperature $u \to \infty$, which doesn't make sense. Set $\lambda = \mu^2 > 0$. Then $X''(x) - \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 - \mu^2 = 0$, which has roots $r = \pm\mu$.Thus $X(x) = C_1e^{\mu x} + C_2e^{-\mu x}$. We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \Rightarrow C_1 + C_2 = 0 \\ X(2) &= 0 \Rightarrow C_1e^{2\mu} + C_2e^{-2\mu} = 0. \end{aligned}$$

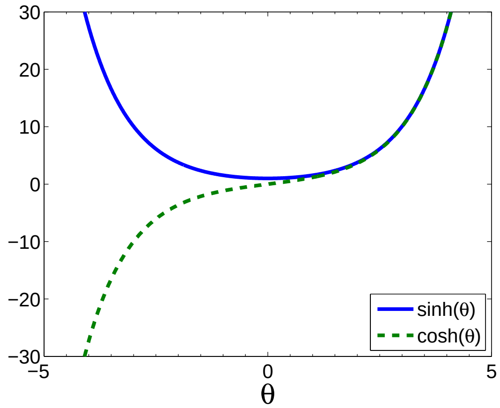

Figure 1: Hyperbolic functions $\sinh(\theta)$ and $\cosh(\theta)$.

Solving simultaneously we find $C_1 = C_2 = 0$. (The first equation gives $C_2 = -C_1$, plugging into the first equation gives $C_1 e^{2\mu} - C_1 e^{-2\mu} = 0 \Rightarrow C_1(e^{2\mu} - e^{-2\mu}) = 0$, and this means that $C_1 = 0$ because $e^{2\mu} - e^{-2\mu}$is only zero at $\mu = 0$, which it isn't here - $\mu^2 = \lambda > 0$). Thus we have recovered the trivial solution (aka zero solution). Therefore for $\lambda > 0$ we have no eigenvalues or eigenfunctions. As we had expected.

ASIDE

It is actually more convenient to use hyperbolic functions, and write down $X(x) = \tilde{C}_1 \sinh(\mu x) + \tilde{C}_2 \cosh(\mu x)$,because they have some nice properties. Recall

$$\sinh(\theta) = \frac{e^\theta - e^{-\theta}}{2}, \ \cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2}.$$

See Fig.1 for an illustration. The convenient properties include that $\sinh(\theta) = 0$ ONLY at $\theta = 0$, $\cosh(\theta) = 0$ NEVER, and $\cosh(0) = 1$. So when applying the boundary conditions $X(0) = X(2) = 0$:

$$X(0) = 0 \Rightarrow \ \tilde{C}_1 \sinh(0) + \tilde{C}_2 \cosh(0) = 0 \ \Rightarrow \tilde{C}_2 = 0,$$

since $\sinh(0) = 0$, $\cosh(0) = 1$. Then $X(x) = \tilde{C}_1 \sinh(\mu x)$;

$$X(2) = 0 \Rightarrow \ X(2) = \tilde{C}_1 \sinh(2\mu) \ \Rightarrow \tilde{C}_1 = 0.$$

This is the case because $\sinh(2\mu)$ is only zero at $\mu = 0$, but μ is non-zero by definition.

END OF ASIDE

All right, next we consider the $\lambda = 0$ case (we could consider it jointly with the $\lambda < 0$ or $\lambda > 0$ cases, if we're very careful, but for the purposes of a systematic approach we won't here). Then $X'' = 0 \Rightarrow X(x) = Ax + B$. Applying boundary conditions, $0 = X(0) = B \Rightarrow B = 0$; $0 = X(2) = 2A \Rightarrow A = 0$. Thus we have recovered the trivial solution (aka zero solution). Therefore for $\lambda = 0$ we have no eigenvalues or eigenfunctions.

Finally we look at the $\lambda < 0$ case. Set $\lambda = -\mu^2 < 0$. Then $X''(x) + \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 + \mu^2 = 0$, which has roots $r = \pm i\mu$.Thus $X(x) = \tilde{C}_1 e^{i\mu x} + \tilde{C}_2 e^{-i\mu x}$ or $X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$ (for more details on solving this ode, see your textbook, section 3.3). We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \ \Rightarrow \ C_1 \sin(0) + C_2 \cos(0) = 0 \ \Rightarrow \ C_2 = 0 \\ X(2) &= 0 \ \Rightarrow \ C_1 \sin(2\mu) = 0. \end{aligned}$$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \dots$, the second condition tells us that $2\mu = n\pi$ or $\mu = n\pi/2$, $n = 1, 2, 3, \dots$ Thus we have our eigenfunctions an eigenvalues for $\lambda < 0$:

$$\begin{aligned} \lambda_n &= -\left(\frac{n\pi}{2}\right)^2 \\ X_n(x) &= \sin(n\pi x/2). \end{aligned}$$

Now we re-assemble. Recall $u(x,t) = X(x)T(t)$. Therefore

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{2}\right)\exp\left(-\frac{n^2\pi^2}{16}t\right)$$

for $n = 1, 2, 3, \ldots$ are each solutions to the pde. The pde is linear so we can use the principle of superposition, and sum them to make up a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)\exp\left(-\frac{n^2\pi^2}{16}t\right),$$

where the b_n are constants.

We solve for the b_n using the initial condition. That is, $u(x,0) = f(x)$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right),$$

which is a Fourier sine series. We exploit orthogonality of the sines, that is,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

where $L = 2$ to solve for the individual b_n:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

since $L = 2$. Now here $f(x) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)$, so

$$b_n = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n = 2\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx - \int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx + 4\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx.$$

This is less scary than it looks! We can use orthogonality of the sines:

$$\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 1, & n = 4 \\ 0, & \text{otherwise} \end{cases}$$

So then

$$b_1 = 2\int_0^2 \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx - 0 + 0 = 2$$

$$b_2 = 0 - \int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx + 0 = -1$$

$$b_4 = 0 - 0 + 4\int_0^2 \sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx = 4$$

$$b_n = 0 \text{ if } n \neq 1, 2, 4.$$

Thus $b_1 = 2$,$b_2 = -1$, $b_4 = 4$, and $b_n = 0$ for all other values of n. We can now re-write our solution:

$$u(x,t) = 2\sin\left(\frac{\pi x}{2}\right)\exp\left(-\frac{\pi^2}{16}t\right) - \sin(\pi x)\exp\left(-\frac{\pi^2}{4}t\right) - \sin(2\pi x)\exp\left(-\pi^2 t\right).$$

2. Find the solution to the heat conduction problem:

$$\begin{aligned} u_t &= \alpha^2 u_{xx},\ 0 \le x \le \pi,\ t > 0 \\ u(0,t) &= 0 \\ u_x(\pi,t) &= 0 \\ u(x,0) &= 3\sin\left(\frac{5x}{2}\right) = f(x) \end{aligned}$$

The mechanics of this problem are very VERY similar to the previous problem. In fact, that's the case with most of the heat equation problems. The main difference is in the eigenvalue/eigenfunction part!

Solution:

We use separation of variables. Let $u(x,t) = X(x)T(t)$. Then $u_t = \alpha^2 u_{xx}$ becomes $X(x)T'(t) = \alpha^2 X''(x)T(t)$. We divide both sides by $\alpha^2 X(x)T(t)$ to obtain:

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = \lambda, \tag{2}$$

where λ is a constant.

What happens to the boundary conditions under the separation of variables?

$$0 = u(0,t) = X(0)T(t) \Rightarrow X(0) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

$$0 = u_x(\pi,t) = X'(\pi)T(t) \Rightarrow X'(\pi) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

So we have $X(0) = X'(\pi) = 0$.

Let's start with the T-equation from (3):

$$T'(t) = \lambda\alpha^2 T(t),$$

which means $T(t) = Ce^{\alpha^2\lambda t}$ (see above for details on solving this ODE).

Next we deal with the X-equation in (3) with conditions $X(0) = X'(\pi) = 0$ derived from the boundary conditions

$$\begin{aligned} X'' &= \lambda X \\ X(0) &= X'(\pi) \ = \ 0. \end{aligned}$$

This is an *eigenvalue problem.* There are 3 cases to consider: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

We begin with the $\lambda > 0$ case - recall from above that we expect this to only yield the trivial solution (aka $X = 0$). Set $\lambda = \mu^2 > 0$. Then $X''(x) - \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 - \mu^2 = 0$, which has roots $r = \pm\mu$.Thus $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \ \Rightarrow \ C_1 + C_2 = 0 \\ X'(\pi) &= 0 \ \Rightarrow \ \mu C_1 e^{\pi\mu} - \mu C_2 e^{-\pi\mu} = 0. \end{aligned}$$

Solving simultaneously we find $C_1 = C_2 = 0$. (The first equation gives $C_2 = -C_1$, plugging into the first equation gives $C_1\mu e^{\pi\mu} + C_1\mu e^{-\pi\mu} = 0 \Rightarrow C_1\mu(e^{\pi\mu} + e^{-\pi\mu}) = 0$, and this means that $C_1 = 0$ because $e^{\pi\mu} + e^{-\pi\mu}$ is never zero. You could also use $X(x) = \tilde{C_1}\sinh(\mu x) + \tilde{C_2}\cosh(\mu x)$, and would find $\tilde{C_1} = \tilde{C_2} = 0$.

All right, next we consider the $\lambda = 0$ case (we could consider it jointly with the $\lambda < 0$ or $\lambda > 0$ cases, if we're very careful, but for the purposes of a systematic approach we won't here). Then $X'' = 0 \Rightarrow X(x) = Ax + B$. Applying boundary conditions, $0 = X(0) = B \Rightarrow B = 0$; $0 = X'(\pi) = A \Rightarrow A = 0$. Thus we have recovered the trivial solution (aka zero solution). Therefore for $\lambda = 0$ we have no eigenvalues or eigenfunctions.

Finally we look at the $\lambda < 0$ case. Set $\lambda = -\mu^2 < 0$. Then $X''(x) + \mu^2 X(x) = 0$ and $X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$ (key steps to solving that ode above; for more details see your textbook, section 3.3). We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X(0) &= 0 \Rightarrow C_1 \sin(0) + C_2 \cos(0) = 0 \Rightarrow C_2 = 0 \\ X'(\pi) &= 0 \Rightarrow \mu C_1 \cos(\pi\mu) = 0. \end{aligned}$$

Since $\cos(\theta)$ has roots at $\theta = (2n-1)\pi/2$, $n = 1, 2, 3, \ldots$ (or, equivalently, that $\theta = (2n+1)\pi/2$, $n = 0, 1, 2, 3, \ldots$), the second condition tells us that $\pi\mu = (2n-1)\pi/2$ or $\mu = (2n-1)/2 = (n-1/2)$, $n = 1, 2, 3, \ldots$ Thus we have our eigenfunctions an eigenvalues for $\lambda < 0$:

$$\begin{aligned} \lambda_n &= -\left(n - \frac{1}{2}\right)^2 \\ X_n(x) &= \sin\left((n - 1/2)x\right). \end{aligned}$$

Now we re-assemble. Recall $u(x,t) = X(x)T(t)$. Therefore

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\left(n - \frac{1}{2}\right)x\right) \exp\left(-\alpha^2 \left(n - \frac{1}{2}\right)^2 t\right)$$

for $n = 1, 2, 3, \ldots$ are each solutions to the pde. The pde is linear so we can use the principle of superposition, and sum them to make up a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\left(n - \frac{1}{2}\right)x\right) \exp\left(-\alpha^2 \left(n - \frac{1}{2}\right)^2 t\right),$$

where the b_n are constants.

We solve for the b_n using the initial condition. That is, $u(x,0) = f(x)$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\left(n - \frac{1}{2}\right)x\right),$$

which is a Fourier sine series. As discussed above,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx = \int_0^\pi f(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx$$

which we find by exploiting the orthogonality of sines:

$$\int_0^\pi \sin\left(\left(n - \frac{1}{2}\right)x\right) \sin\left(\left(m - \frac{1}{2}\right)x\right) dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}.$$

Now here $f(x) = 3\sin(5x/2)$, so

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx = \frac{6}{\pi}\int_0^\pi \sin\left(\frac{5x}{2}\right) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx.$$

This is less scary than it looks! We can use orthogonality of the sines:

$$\int_0^\pi \sin\left(\frac{5x}{2}\right) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx = \begin{cases} \pi/2, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

So then

$$b_n = \frac{6}{\pi}\int_0^\pi \sin\left(\frac{5x}{2}\right) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx = \begin{cases} 3, & n = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Thus $b_3 = 3\pi/2$, $b_n = 0$ for $n \neq 3$. We could also just read it off:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\left(n - \frac{1}{2}\right)x\right) \\ 3\sin\left(\frac{5x}{2}\right) &= b_1 \sin\left(\frac{x}{2}\right) + b_2 \sin\left(\frac{3x}{2}\right) + b_3 \sin\left(\frac{5x}{2}\right) + b_4 \sin\left(\frac{7x}{2}\right) + b_5 \sin\left(\frac{9x}{2}\right) + \ldots, \end{aligned}$$

we can see quite plainly that $b_3 = 3$, $b_n = 0$ for $n \neq 3$.

Thus we have found our solution:

$$u(x,t) = 3\sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right).$$

Note that we can always check our solutions by plugging them back into the pde!

$$\begin{aligned} u_t(x,t) &= -\frac{75}{4}\alpha^2 \sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right) \\ u_{xx}(x,t) &= -\frac{75}{4}\sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right) \end{aligned}$$

and then

$$u_t = \alpha^2 u_{xx} \Rightarrow$$

$$-\frac{75}{4}\alpha^2 \sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right) = \alpha^2\left(-\frac{75}{4}\sin\left(\frac{5x}{2}\right)\exp\left(-\left(\frac{5\alpha}{2}\right)^2 t\right)\right);$$

check! The solution is correct.

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3. Find the solution to the heat conduction problem:

$$\begin{aligned} u_t &= u_{xx},\ 0 \leq x \leq 2\pi,\ t > 0 \\ u_x(0,t) &= 0 \\ u_x(2\pi,t) &= 0 \\ u(x,0) &= x = f(x) \end{aligned}$$

Again, the mechanics of this problem are very VERY similar to the previous problem. In fact, that's the case with most of the heat equation problems. The main difference is in the eigenvalue/eigenfunction part! Note that this time around we have Neumann boundary conditions (the boundary conditions are on the spatial derivative of the function; think of this as insulation, no heat flow in or out).

Solution:

We use separation of variables. Let $u(x,t) = X(x)T(t)$. Then $u_t = u_{xx}$ becomes $X(x)T'(t) = X''(x)T(t)$. We divide both sides by $X(x)T(t)$ to obtain:

$$\frac{T'}{T} = \frac{X''}{X} = \lambda, \tag{3}$$

where λ is a constant.

What happens to the boundary conditions under the separation of variables?

$$0 = u(0,t) = X'(0)T(t) \Rightarrow X'(0) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

$$0 = u_x(\pi,t) = X'(2\pi)T(t) \Rightarrow X'(2\pi) = 0 \text{ (since } T(t) \text{ won't be 0 for all} t)$$

So we have $X'(0) = X'(2\pi) = 0$.

Let's start with the T-equation from (3):

$$T'(t) = \lambda T(t),$$

which means $T(t) = Ce^{\lambda t}$ (see above for details on solving this ODE).

Next we deal with the X-equation in (3) with conditions $X'(0) = X'(2\pi) = 0$ derived from the boundary conditions

$$\begin{aligned} X'' &= \lambda X \\ X'(0) &= X'(2\pi) \; = \; 0. \end{aligned}$$

This is an *eigenvalue problem.* There are 3 cases to consider: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

We begin with the $\lambda > 0$ case - recall from above that we expect this to only yield the trivial solution (aka $X = 0$). Set $\lambda = \mu^2 > 0$. Then $X''(x) - \mu^2 X(x) = 0$. Use the substitution $X(x) = e^{rx}$ to get the characteristic equation $r^2 - \mu^2 = 0$, which has roots $r = \pm\mu$.Thus $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X'(0) &= 0 \;\Rightarrow\; \mu C_1 - \mu C_2 = 0 \\ X'(2\pi) &= 0 \;\Rightarrow\; \mu C_1 e^{2\pi\mu} - \mu C_2 e^{-2\pi\mu} = 0. \end{aligned}$$

Solving simultaneously we find $C_1 = C_2 = 0$. (The first equation gives $C_2 = C_1$, plugging into the first equation gives $C_1 \mu e^{2\pi\mu} - C_1 \mu e^{-2\pi\mu} = 0 \Rightarrow C_1 \mu (e^{2\pi\mu} - e^{-2\pi\mu}) = 0$, and this means that $C_1 = 0$ because $e^{2\pi\mu} - e^{-2\pi\mu}$ is never zero for $\mu \neq 0$ (which it isn't by assumption on λ, $\lambda > 0$). You could also use $X(x) = \tilde{C_1} \sinh(\mu x) + \tilde{C_2} \cosh(\mu x)$, and would find $\tilde{C_1} = \tilde{C_2} = 0$.

All right, next we consider the $\lambda = 0$ case (we could consider it jointly with the $\lambda < 0$ or $\lambda > 0$ cases, if we're very careful, but for the purposes of a systematic approach we won't here). Then $X'' = 0 \Rightarrow X(x) = Ax + B$. Applying boundary conditions, $0 = X'(0) = A \Rightarrow A = 0$; $0 = X'(2\pi) = A \Rightarrow A = 0$. So $X(x) = B$, a constant, is still a possible solution! Therefore we do NOT have a trivial solution; rather we have found that $\lambda_0 = 0$ is an eigenvalue with corresponding eigenfunction $X_0(x) = 1$, a constant (which we will multiply by an arbitrary constant below to give the general solution).

Finally we look at the $\lambda < 0$ case. Set $\lambda = -\mu^2 < 0$. Then $X''(x) + \mu^2 X(x) = 0$ and $X(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x)$ (key steps to solving that ode above; for more details see your textbook, section 3.3). We now use the boundary conditions to find constants such that the conditions are satisfied:

$$\begin{aligned} X'(0) &= 0 \;\Rightarrow\; C_1 \cos(0) + C_2 \sin(0) = 0 \;\Rightarrow\; C_1 = 0 \\ X'(2\pi) &= 0 \;\Rightarrow\; \mu C_2 \sin(2\pi\mu) = 0. \end{aligned}$$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \ldots$, the second condition tells us that $2\pi\mu = n\pi$ or $\mu = n/2$, $n = 1, 2, 3, \ldots$ Thus we have our eigenfunctions an eigenvalues for $\lambda < 0$:

$$\begin{aligned} \lambda_n &= -\left(\frac{n}{2}\right)^2 \\ X_n(x) &= \cos\left(\frac{nx}{2}\right). \end{aligned}$$

Now we re-assemble. Recall $u(x,t) = X(x)T(t)$. Therefore

$$u_n(x,t) = X_n(x) T_n(t) = \cos\left(\frac{nx}{2}\right) \exp\left(-\frac{n^2 t}{4}\right)$$

for $n = 0, 1, 2, 3, \ldots$ are each solutions to the pde. Notice that this includes the $n = 0$ case! $u_0(x,t) = \cos(0)\exp(0) = 1$. The pde is linear so we can use the principle of superposition, and sum them to make up a general solution:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right) \exp\left(-\frac{n^2 t}{4}\right)$$

where the a_n are constants. We have used here the convention of writing the $n = 0$ term as $a_0/2$.

We solve for the b_n using the initial condition. That is, $u(x, 0) = f(x)$ so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right),$$

which is a Fourier cosine series. Similar to the case for sines,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{nx}{2}\right) dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right) dx$$

which we find by exploiting the orthogonality of cosines:

$$\int_0^{2\pi} \cos\left(\frac{mx}{2}\right)\cos\left(\frac{nx}{2}\right) dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 2\pi, & m = n = 0 \end{cases}.$$

What is different from the sine series here (other than the fact that we have cosines) is that we treat the 0-eigenvalue term as well, and that gives a different result. That is for $m = n \neq 0$:

$$\int_0^{2\pi} \cos^2\left(\frac{mx}{2}\right) dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos(mx))\, dx = \frac{1}{2}\left(x - \frac{1}{m}\sin(mx)\right)\Big|_{x=0}^{x=2\pi} = \pi,$$

but for $m = n = 0$:

$$\int_0^{2\pi} dx = 2\pi.$$

So now we integrate with $f(x) = x$ to find a_n. First let's look at a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\, dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_{x=0}^{x=2\pi} = 2\pi.$$

You can also think if this integral as $\frac{1}{\pi}\int_0^{2\pi} x\cos(n * 0/2)\, dx$. Now, for a_n, $n \neq 0$:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos\left(\frac{nx}{2}\right) dx.$$

We integrate by parts; $u = x \Rightarrow du = dx$ and $dv = \cos(nx/2) \Rightarrow v = 2\sin(nx/2)/n$. Then $a_n = 1/\pi(uv - \int vdu)$ or

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\frac{2x}{n}\sin\left(\frac{nx}{2}\right)\Big|_{x=0}^{x=2\pi} - \frac{2}{n}\int_0^{2\pi}\sin\left(\frac{nx}{2}\right) dx\right] \\ &= \frac{1}{\pi}\left[-\frac{2}{n}\left(-\frac{2}{n}\right)\cos\left(\frac{nx}{2}\right)\Big|_{x=0}^{x=2\pi}\right] \\ a_n &= \frac{4}{\pi n^2}\left(\cos(n\pi) - 1\right), \end{aligned}$$

since $\cos(0) = 1$. Now, $\cos(n\pi) = (-1)^n$, so

$$a_n = \frac{4}{n^2\pi}((-1)^n - 1).$$

Notice that this means for n even, $a_n = 0$; for n odd, $a_n = -8/\pi n^2$. We can therefore say:

$$a_{2m} = 0 \text{ and } a_{2m-1} = -\frac{8}{\pi(2m-1)^2}.$$

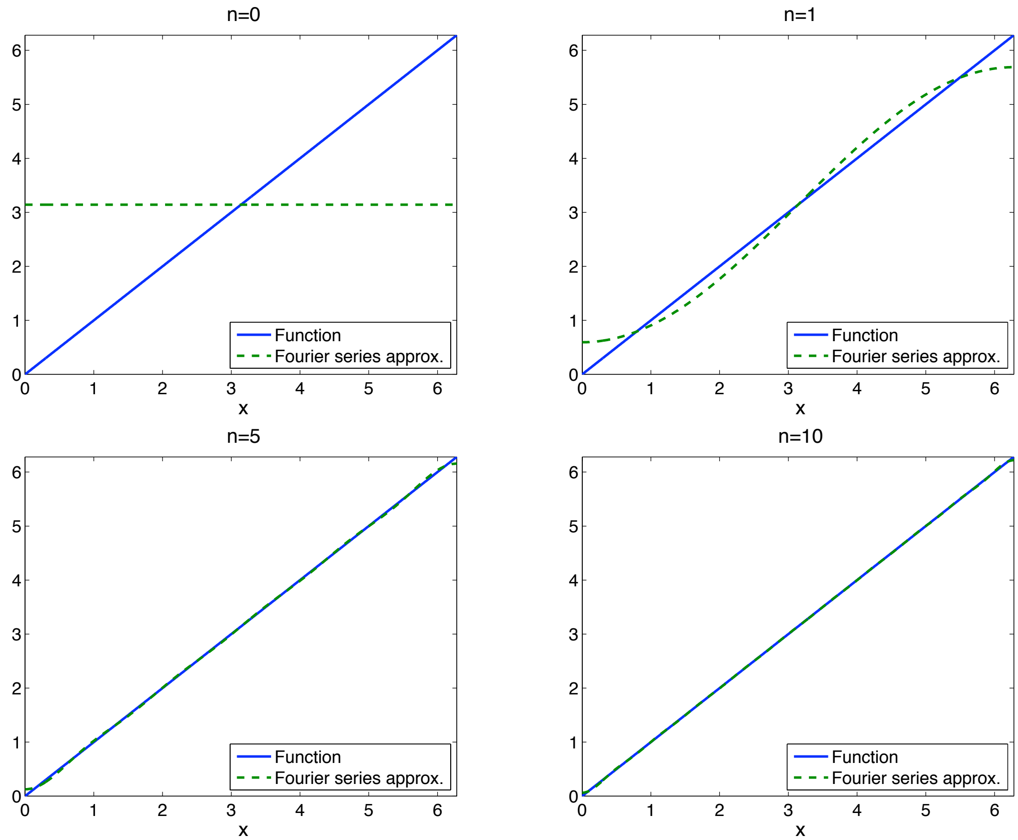

Figure 2: Our Fourier cosine representation of $f(x) = x$ for $n = 0, 1, 5, 10$.

Now let's re-assemble our solution.

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx}{2}\right) \exp\left(-\frac{n^2 t}{4}\right) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_{2m} \cos(mx) \exp\left(-m^2 t\right)$$
$$+ \sum_{m=1}^{\infty} a_{2m-1} \cos\left(\frac{(2m-1)}{2}x\right) \exp\left(-\frac{(2m-1)^2 t}{4}\right),$$

separating out even and odd terms b/c our coefficients are different for evens and odds! In particular, $a_{2m} = 0$. Thus,

$$u(x,t) = \pi - \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos\left(\frac{(2m-1)}{2}x\right) \exp\left(-\frac{(2m-1)^2 t}{4}\right).$$

That's our answer. Notice that as $t \to \infty$, $u \to \pi$, a constant solution, as expected. And that constant is π, which is the *average* initial temperature in the bar, again as expected.

Let's think about how quickly our series solution converges to the actual solution by looking at the Fourier cosine series representation we created of the initial condition. We showed that

$$x = \pi - \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)}{2}x\right), \; 0 \le x \le 2\pi.$$

Fig.2 shows x and the Fourier series version for $m = 0, 1, 5, 10$. Notice that the approximation is quite good even for $n = 5$, and very good for $n = 10$! The series converges most slowly at the endpoints.